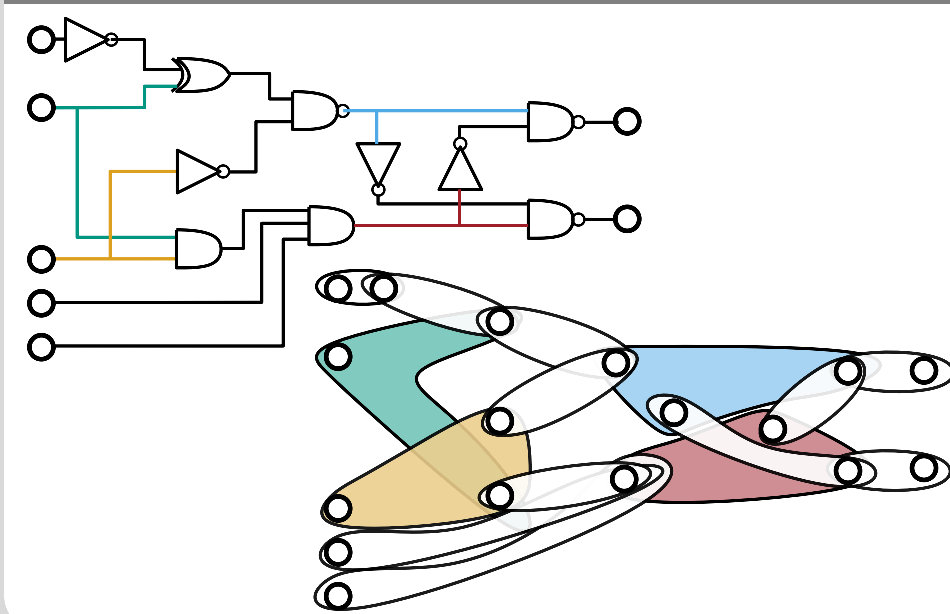
Karlsruhe Institute of Technology

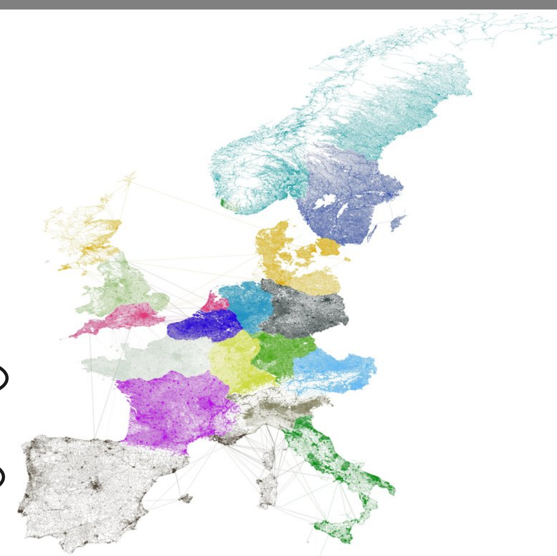
Recent Advances in Graph and Hypergraph Partitioning

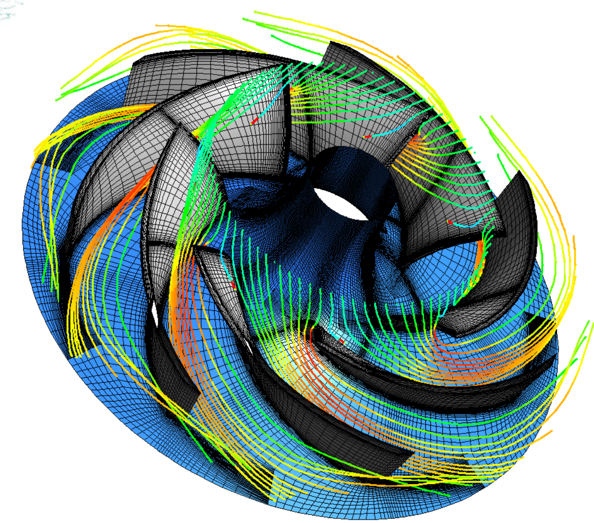
Annual SPP Meeting · October 18, 2017

Yaroslav Akhremtsev, Peter Sanders, Sebastian Schlag, Christian Schulz

INSTITUTE OF THEORETICAL INFORMATICS · ALGORITHMICS GROUP



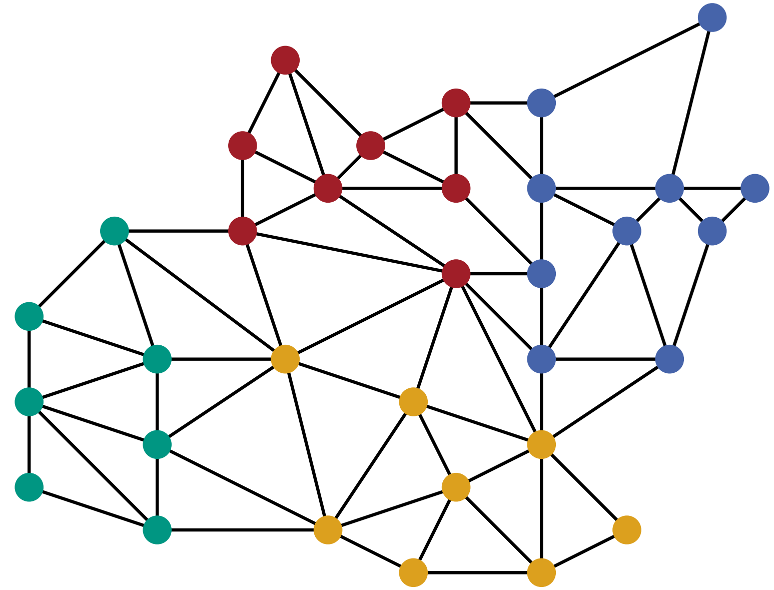




Graphs and Hypergraphs

Graph $G = (V, E)$

vertices → V, edges → E



- models **relationships** between **objects**
- dyadic (**2-ary**) relationships

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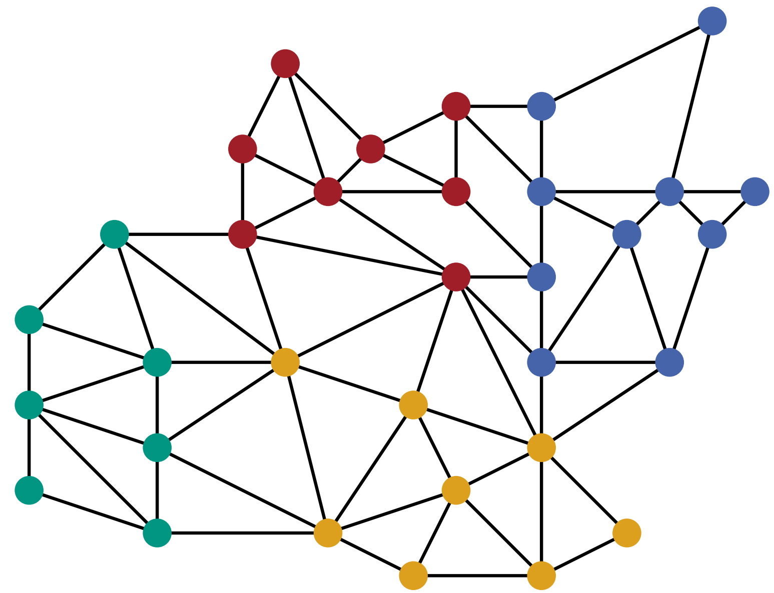
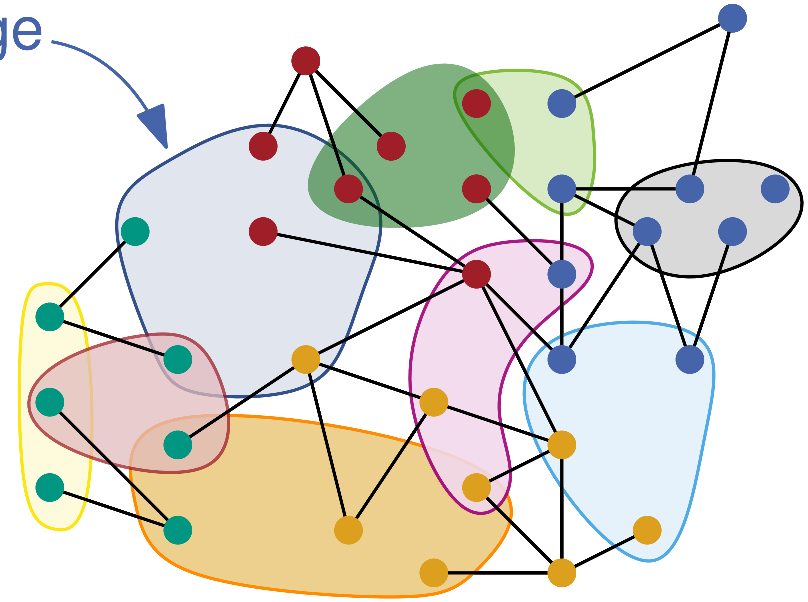


hyperedge



Hypergraph $H = (V, E)$

- Generalization of a graph
  $\Rightarrow$ hyperedges connect ≥ 2 nodes
- arbitrary (**d-ary**) relationships
- Edge set $E \subseteq \mathcal{P}(V) \setminus \emptyset$

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ε-Balanced Graph and Hypergraph Partitioning

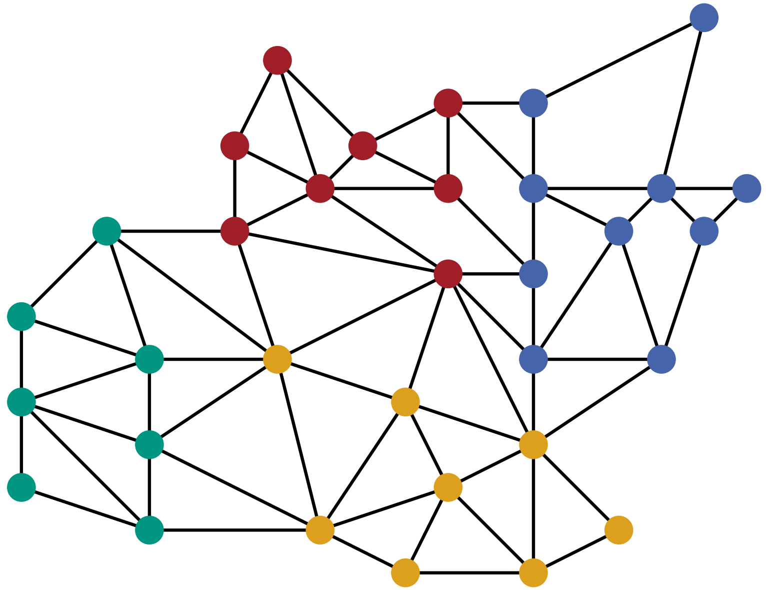
Karlsruhe Institute of Technology

Partition (hyper)graph $G = (V, E, c : V \to \mathbb{R}_{>0}, \omega : E \to \mathbb{R}_{>0})$ into **k** disjoint blocks $V_1, \ldots, V_k$ s.t.

- blocks V_i are **roughly equal-sized**:

$$c(V_i) \leq (1 + \varepsilon) \left\lceil \frac{c(V)}{k} \right\rceil$$

- **objective** function on edges is **minimized**



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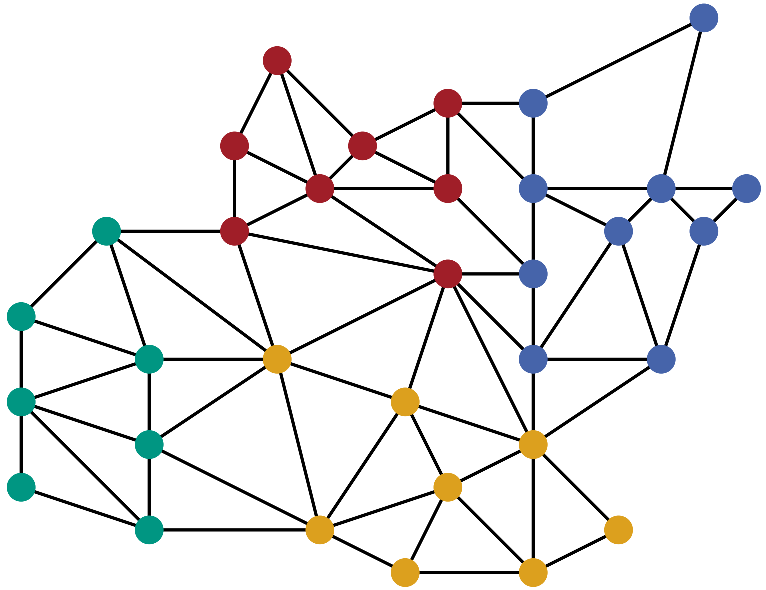

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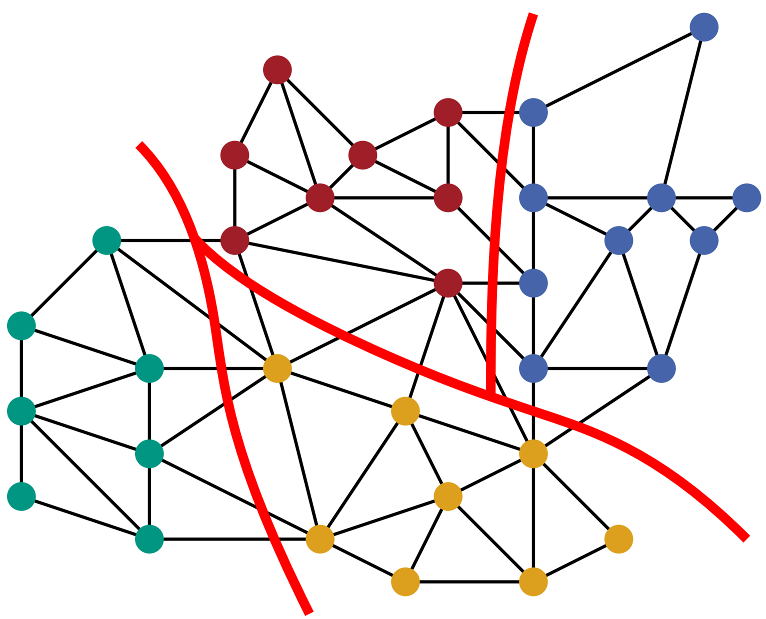
Common Objectives:

- Graphs:
  - cut: $\sum_{e \in \text{cut}} \omega(e)$



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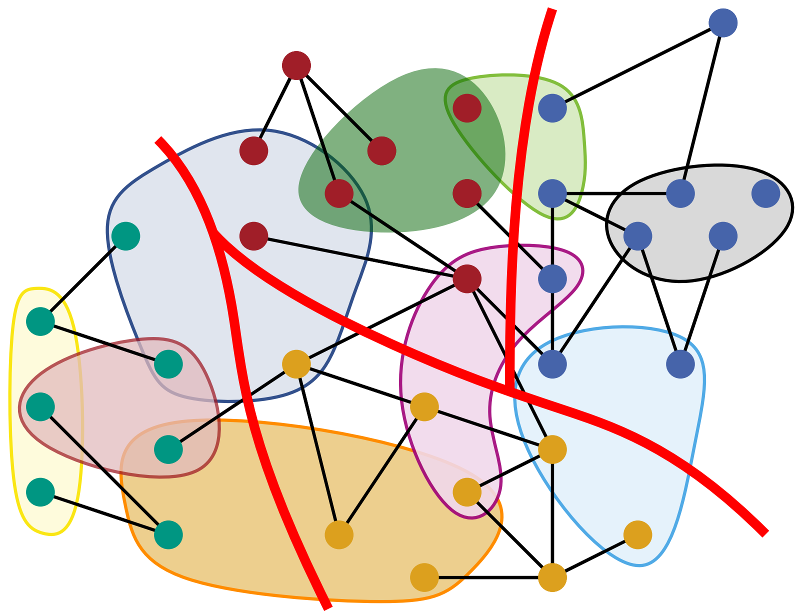

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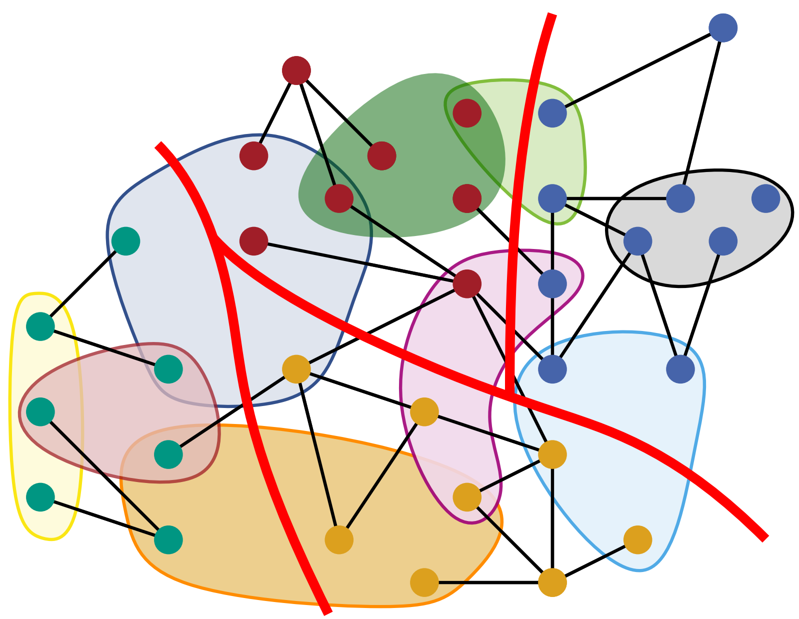

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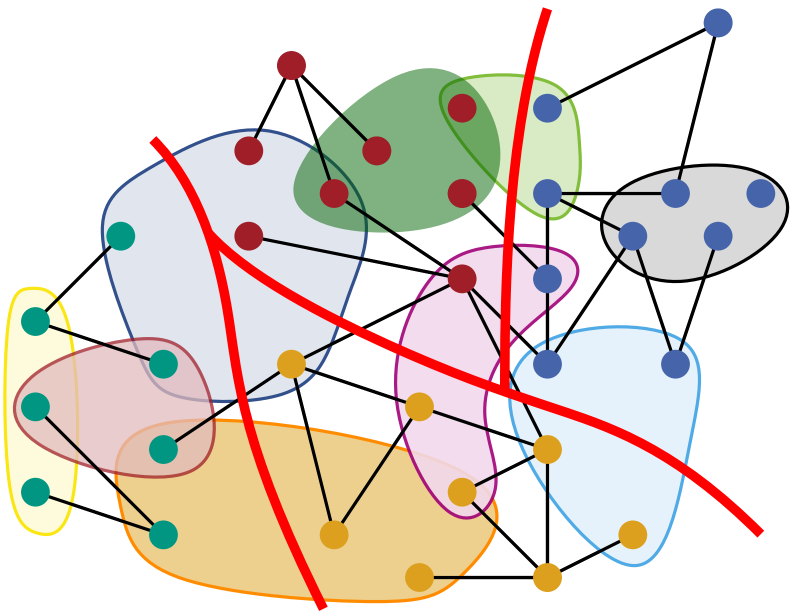
  - connectivity: $\sum_{e \in \text{cut}} (\lambda - 1)\, \omega(e)$



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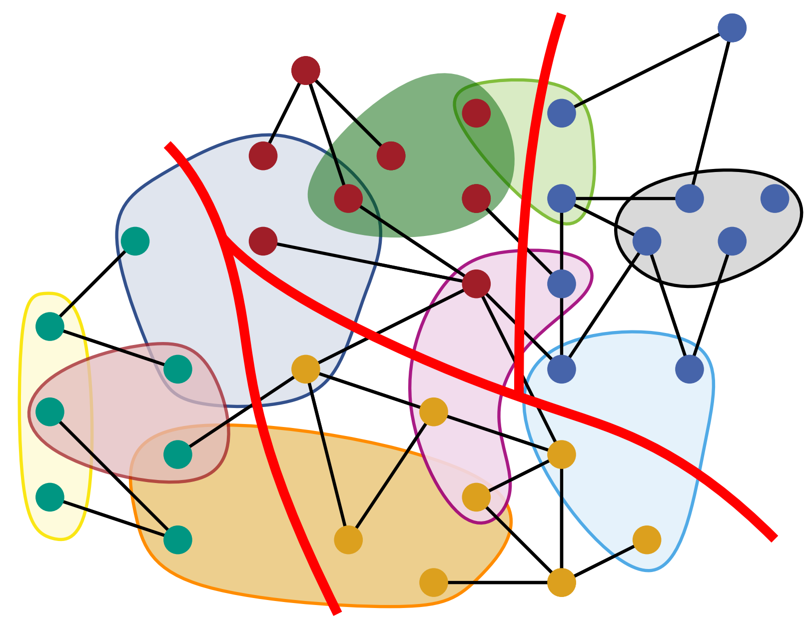
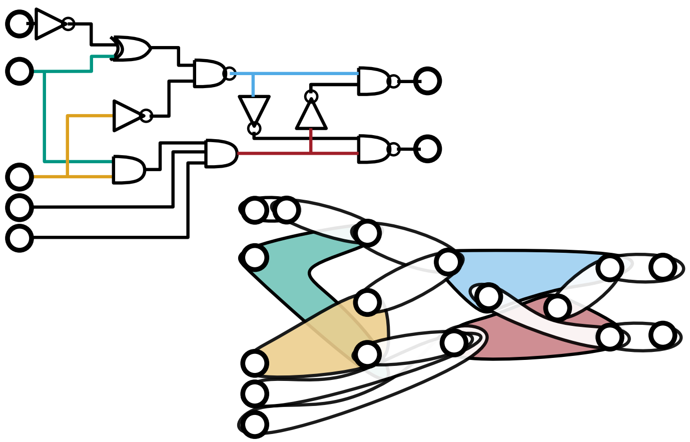
 # blocks connected by e



Applications

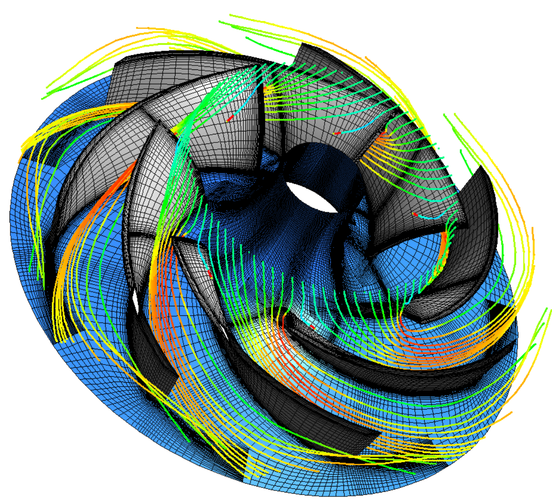


VLSI Design

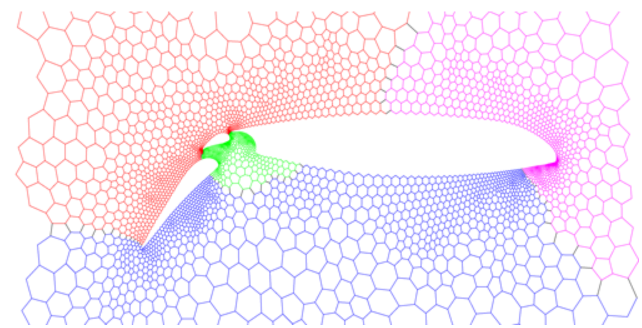
Warehouse Optimization

Complex Networks

Route Planning



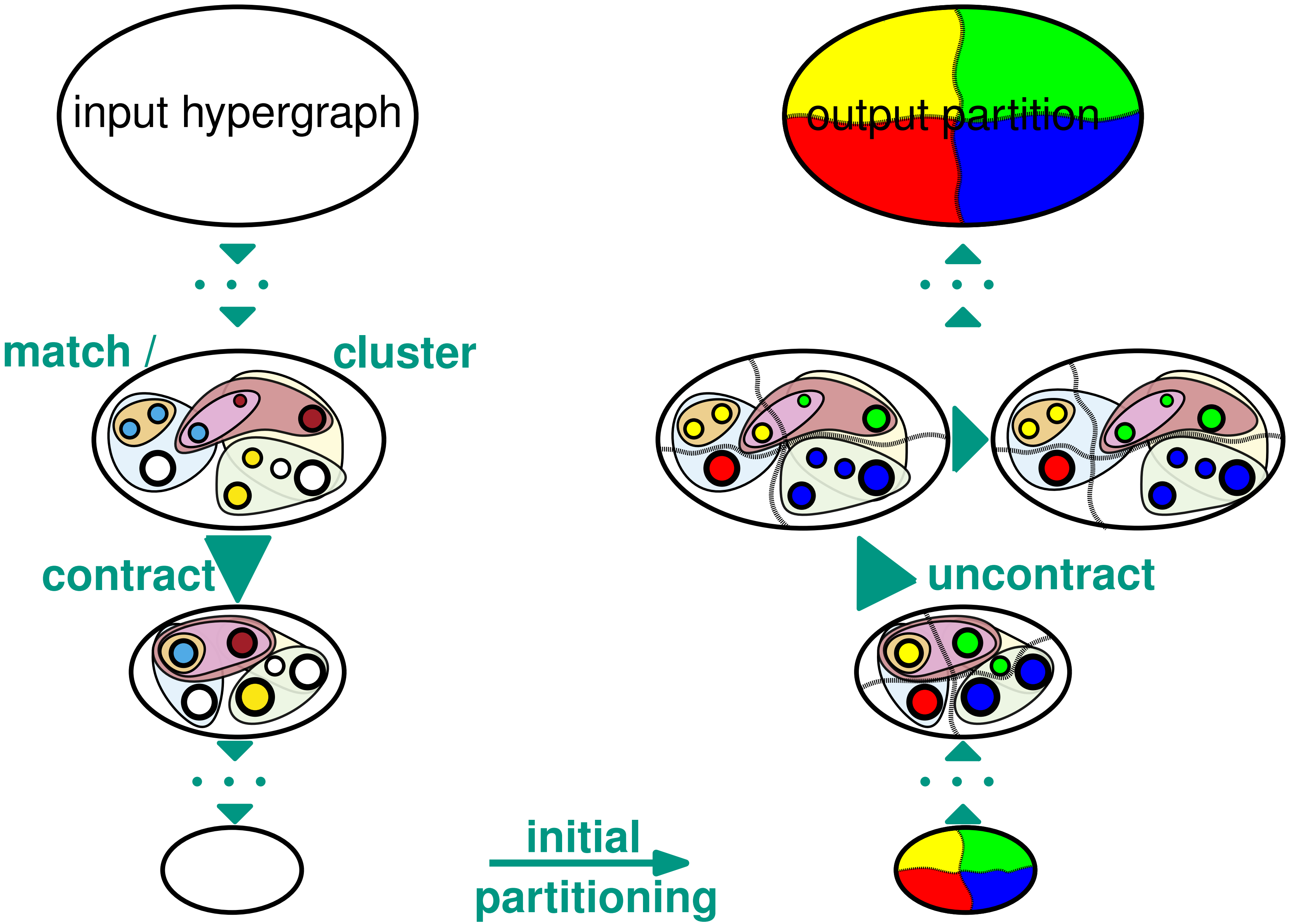
Simulation



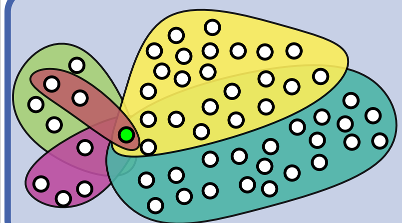
$$\mathbb{R}^{n \times n} \ni Ax = b \in \mathbb{R}^n$$

Scientific Computing

The Multilevel Framework

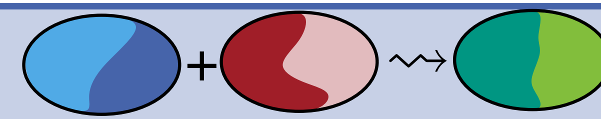


Recent Advances in Hypergraph Partitioning



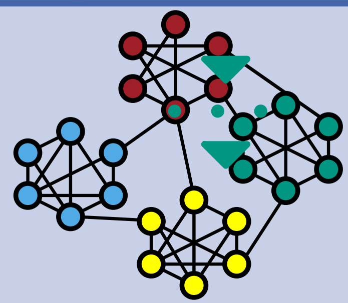
Min-Hash Based Sparsification

Akhremtsev et. al (ALENEX'17)



Memetic Multilevel Algorithm

Andre, Schlag, Schulz (arXiv)



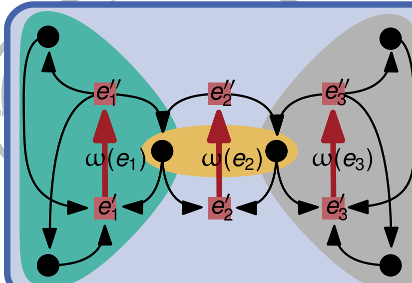
Community-Aware Coarsening

Heuer, Schlag (SEA'17)

Algorithm $A \leftarrow \begin{cases} \text{Config } \mathcal{C}_1 \\ \text{Config } \mathcal{C}_2 \end{cases}$

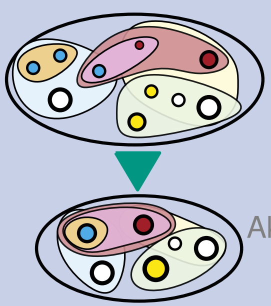
Algorithm Configuration

Öhl, Bachelor's Thesis (upcoming)



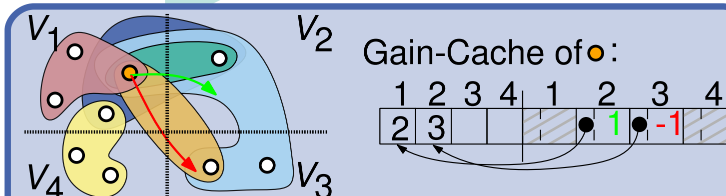
Max-Flow Min-Cut Refinement

Heuer, Master's Thesis (upcoming)



Fast *n*-Level Coarsening

Akhremtsev et. al (ALENEX'17)



Engineered *k*-way FM

Akhremtsev et. al (ALENEX'17)

initial partitioning

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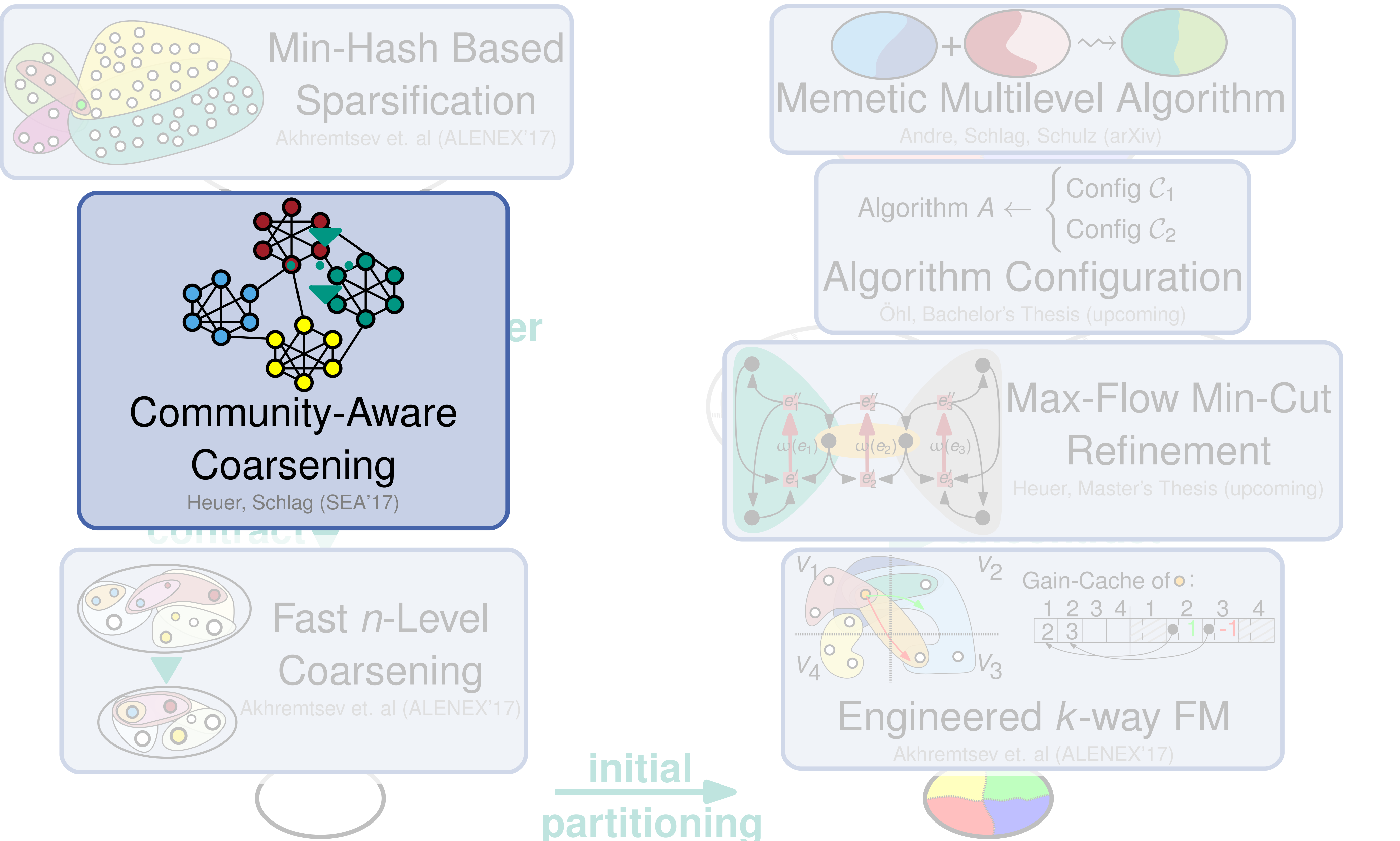

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Clustering-based Coarsening

Common Strategy: **avoid** global decisions ⇝ **local**, greedy algorithms

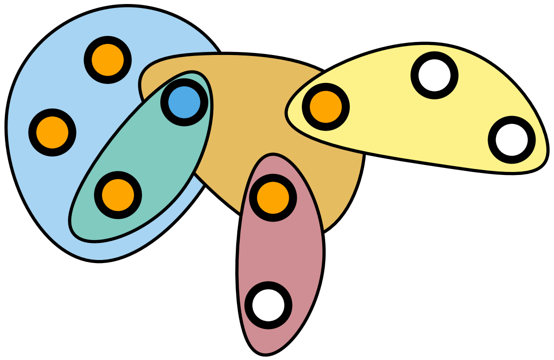
Objective: identify highly connected vertices

using...

foreach *vertex* v **do**

cluster[v] := argmax `rating`(v,u)

neighbor u



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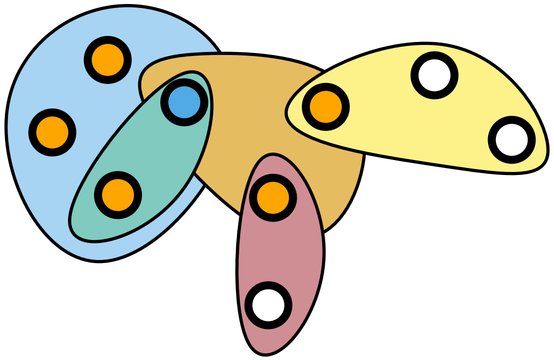

Main Design Goals: [Karypis,Kumar 99]

1: reduce **size** of nets ⇝ easier local search

2: reduce **number** of nets ⇝ easier initial partitioning

3: maintain **structural similarity** ⇝ good coarse solutions

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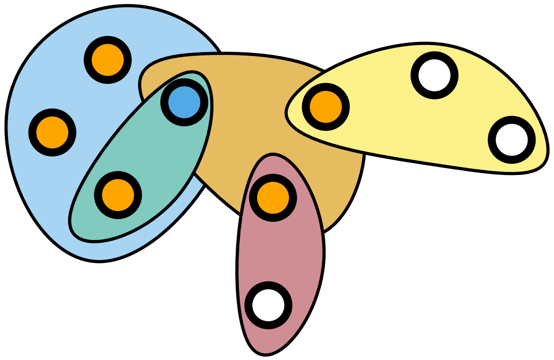

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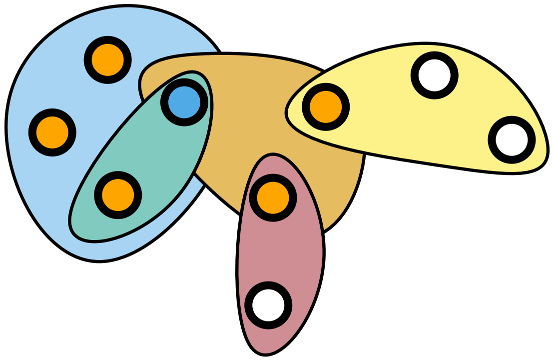

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⟹ ensure ∼balanced vertex weights

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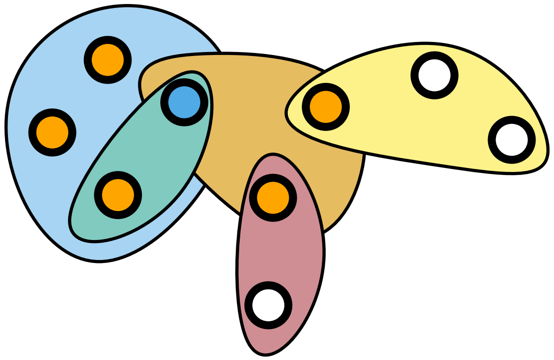

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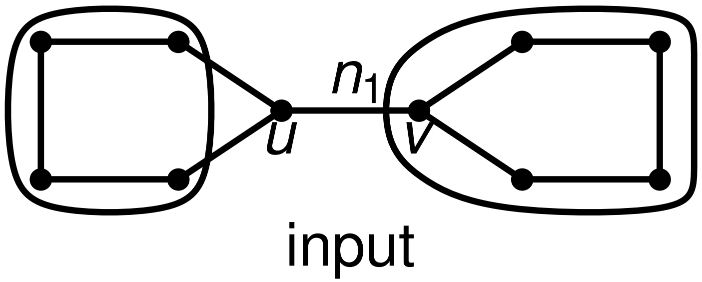
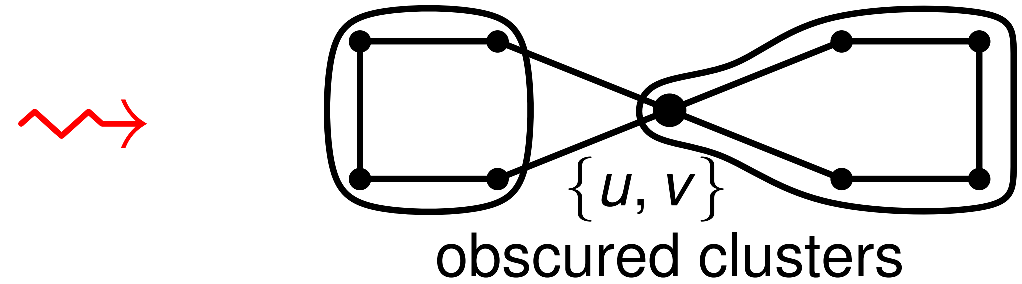
enough?

What could possibly go wrong?

... a lot:





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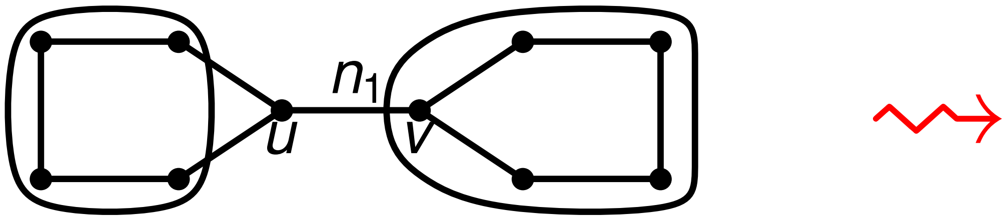
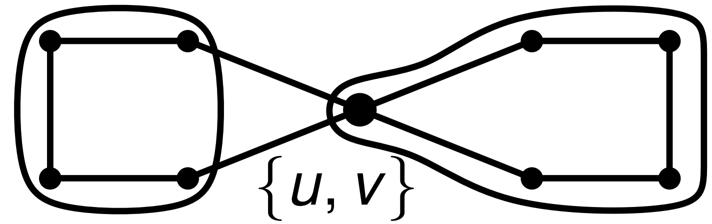
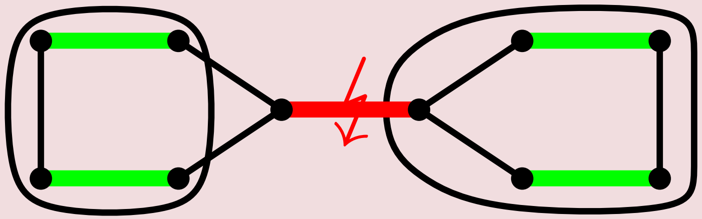


input

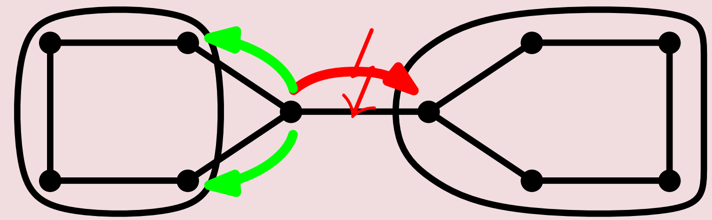


obscured clusters

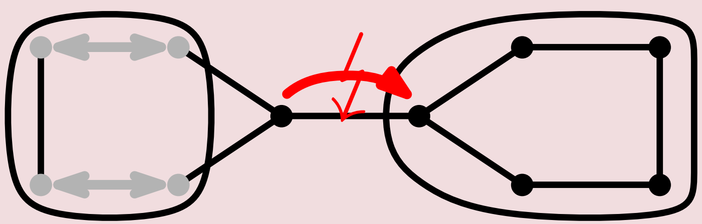
ISSUES



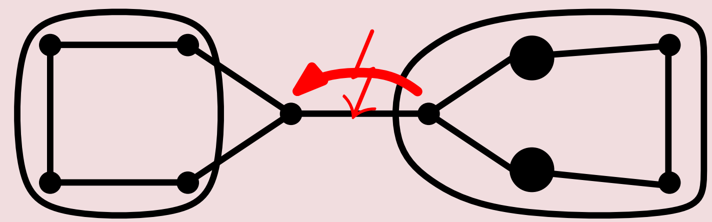
maximal matching



random tie-breaking



prefer unclustered



heavy neighbors

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input

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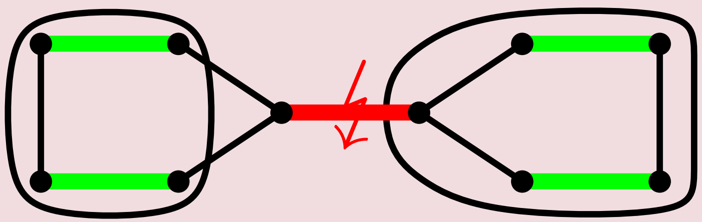

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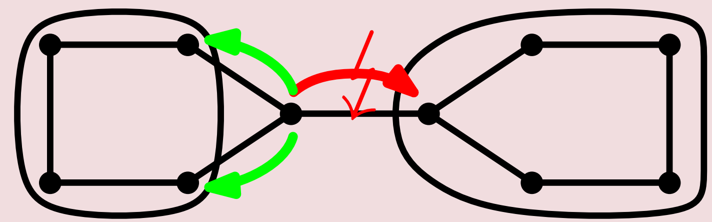

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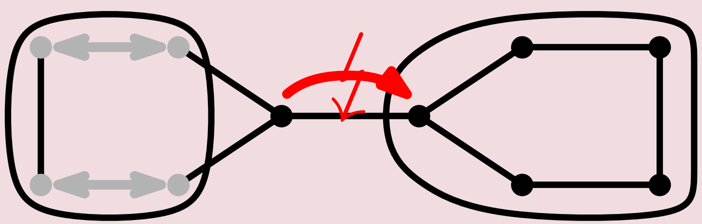

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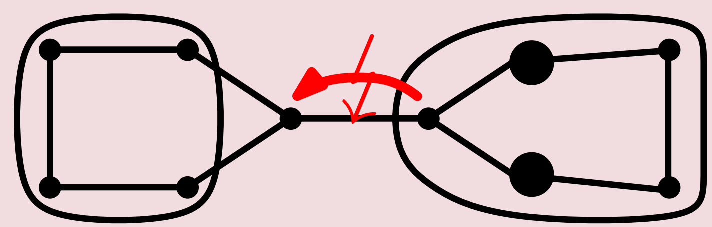

heavy neighbors

⇒ **Problem:** relying **only** on **local** information!

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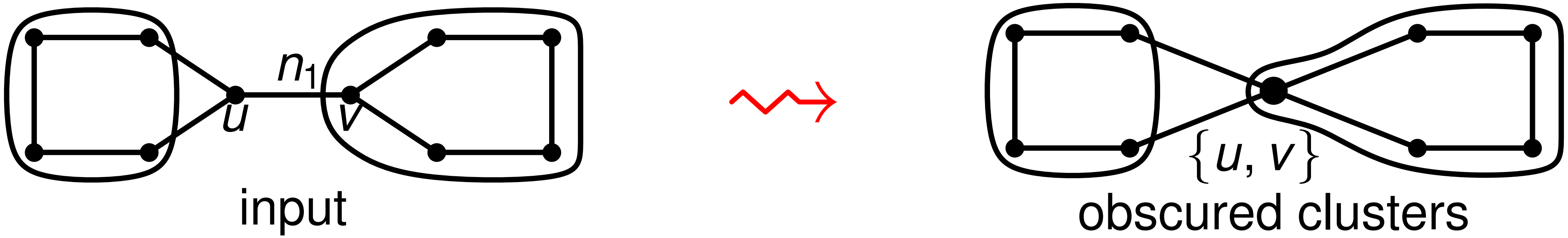
Our Approach: Community-aware Coarsening



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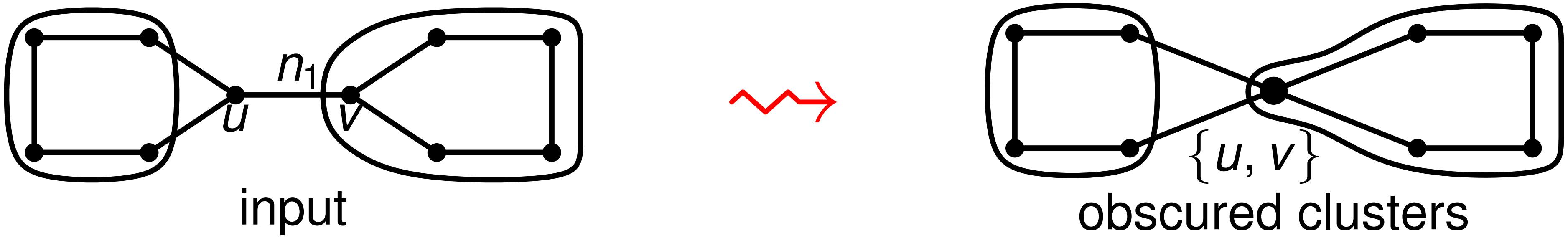
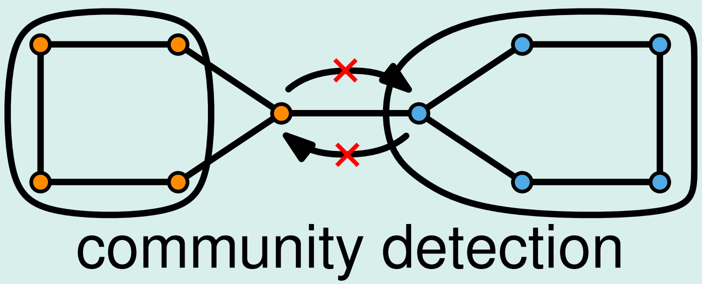


SOLUTION



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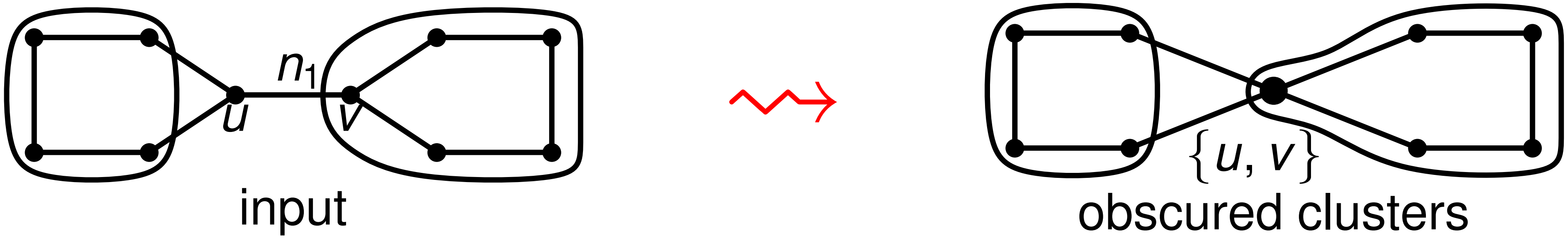

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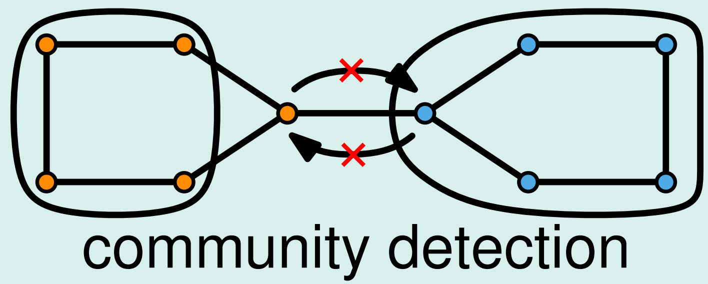
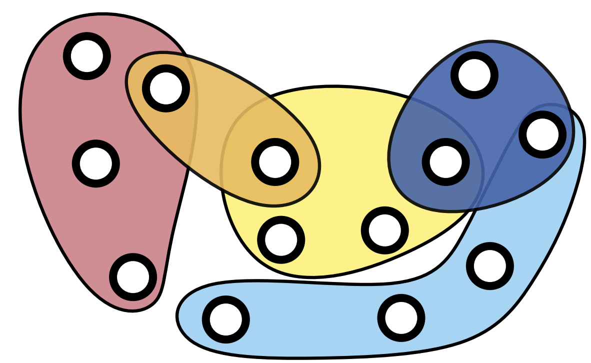


Framework:

- preprocessing: determine **community structure**
  - using **Louvain** algorithm
  - on **bipartite** graph representation
  - structural properties via **edge weights**
- only allow **intra-community** contractions



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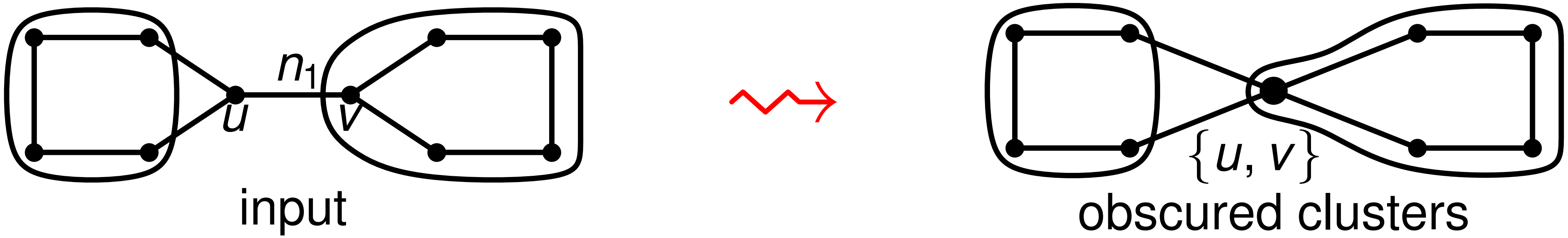

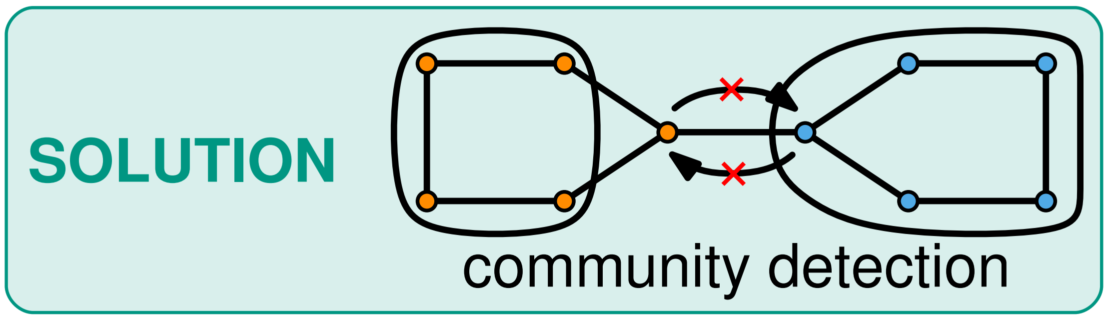

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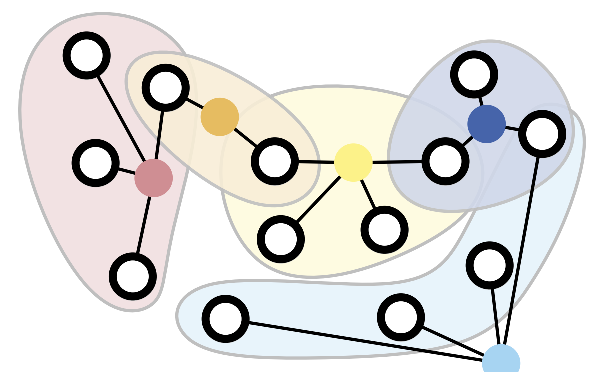

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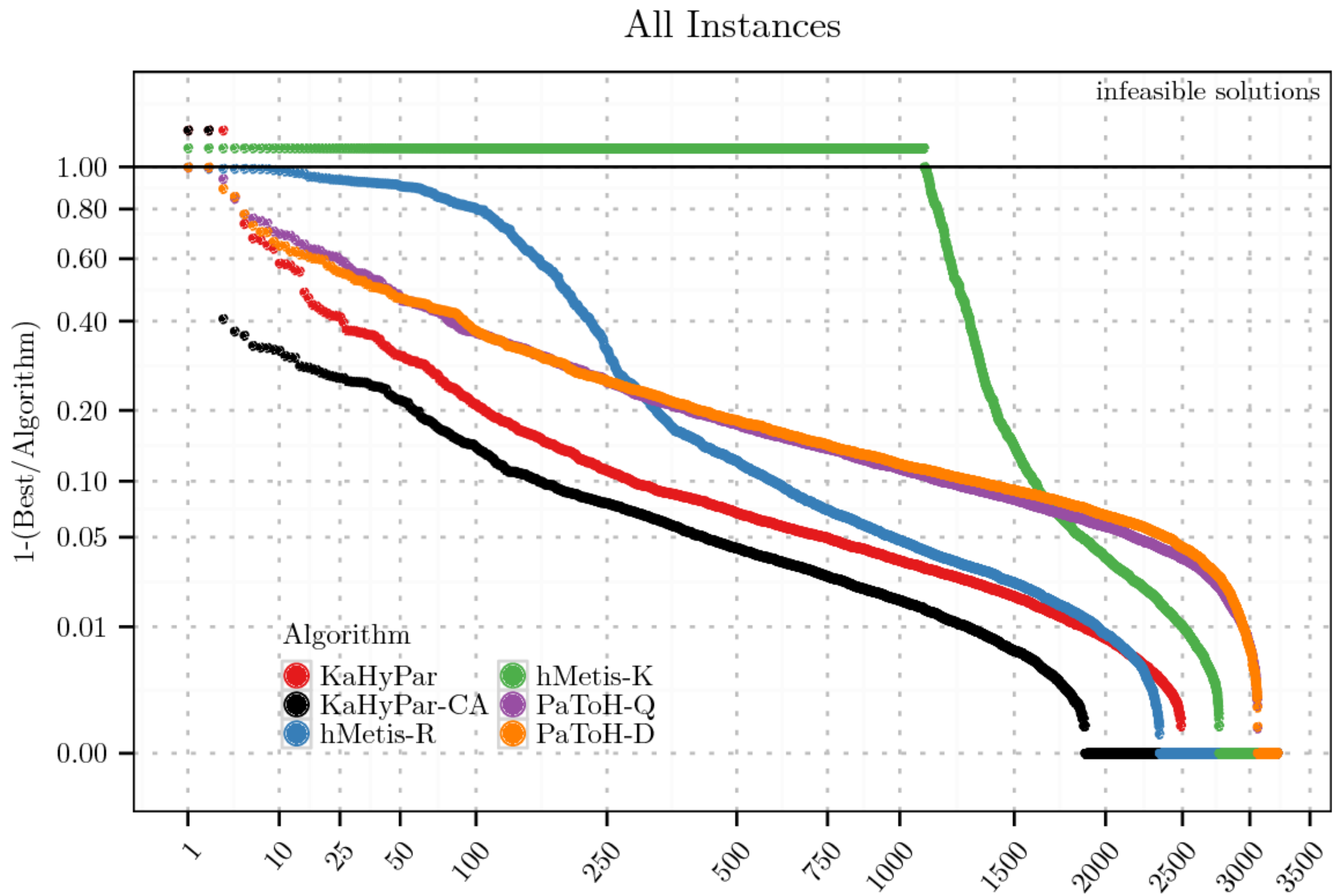
Experimental Results

- 488 hypergraphs (VLSI, UF Sparse Matrix Collection, SAT Competition)
- $k \in \{2, 4, 8, 16, 32, 64, 128\}$ with imbalance: $\varepsilon = 3\%$



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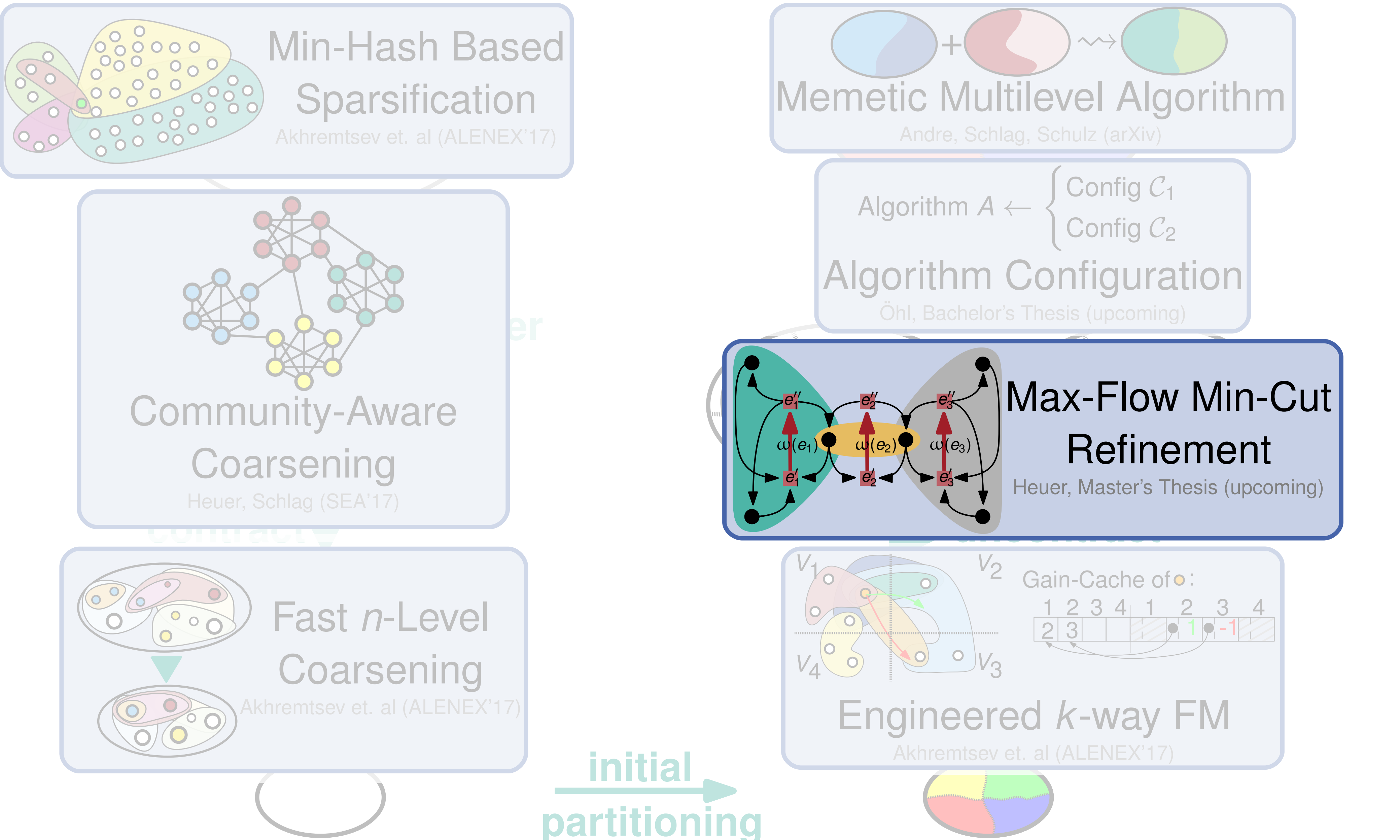
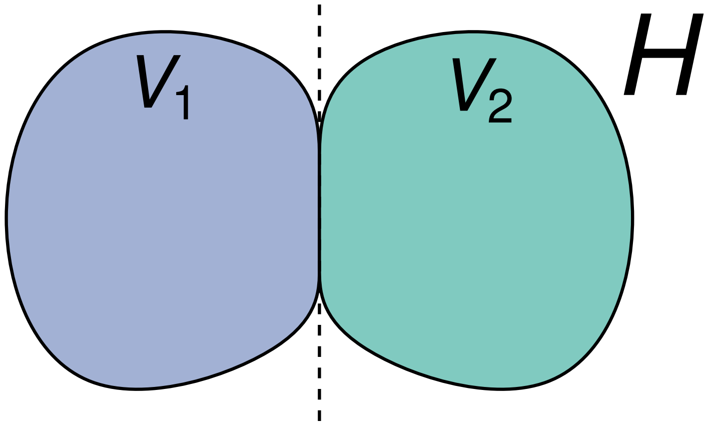
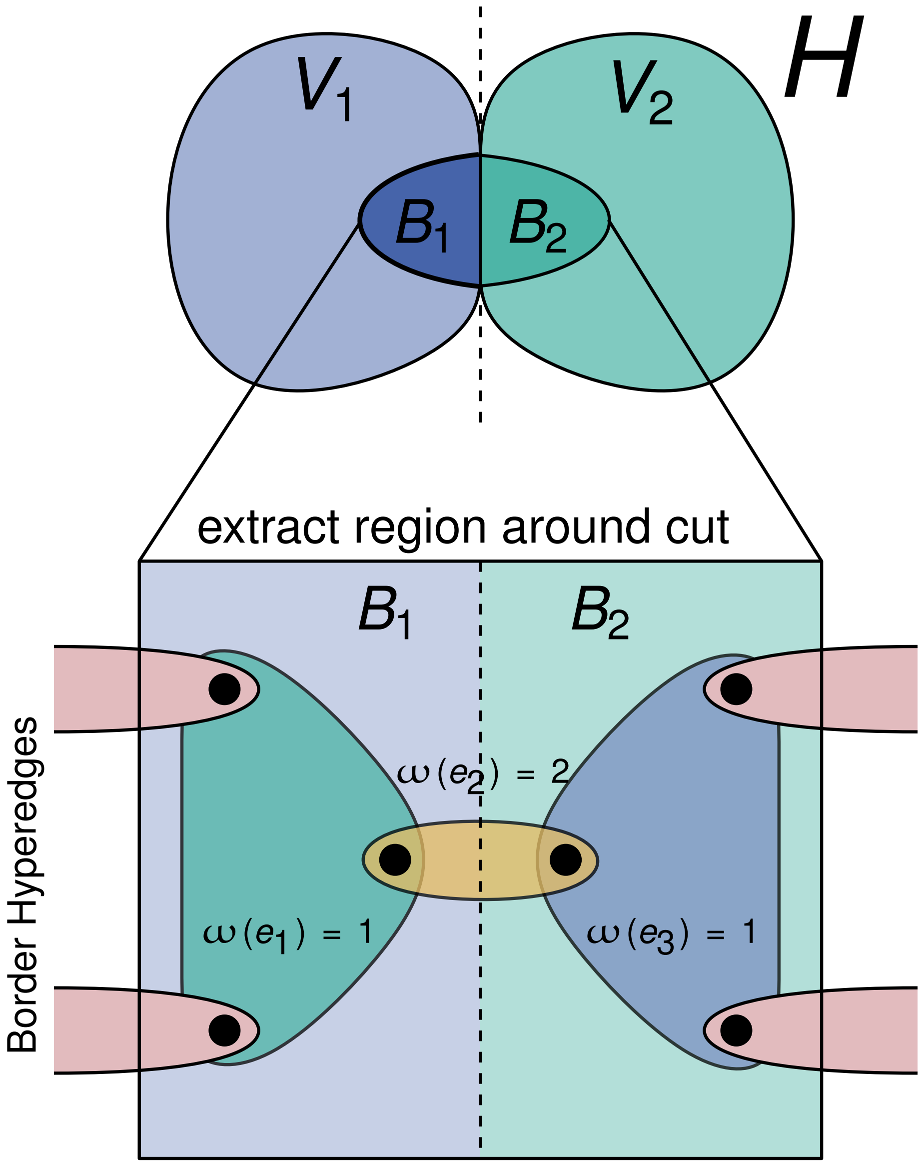
Recent Advances in Hypergraph Partitioning



Max-Flow Min-Cut Refinement



Max-Flow Min-Cut Refinement



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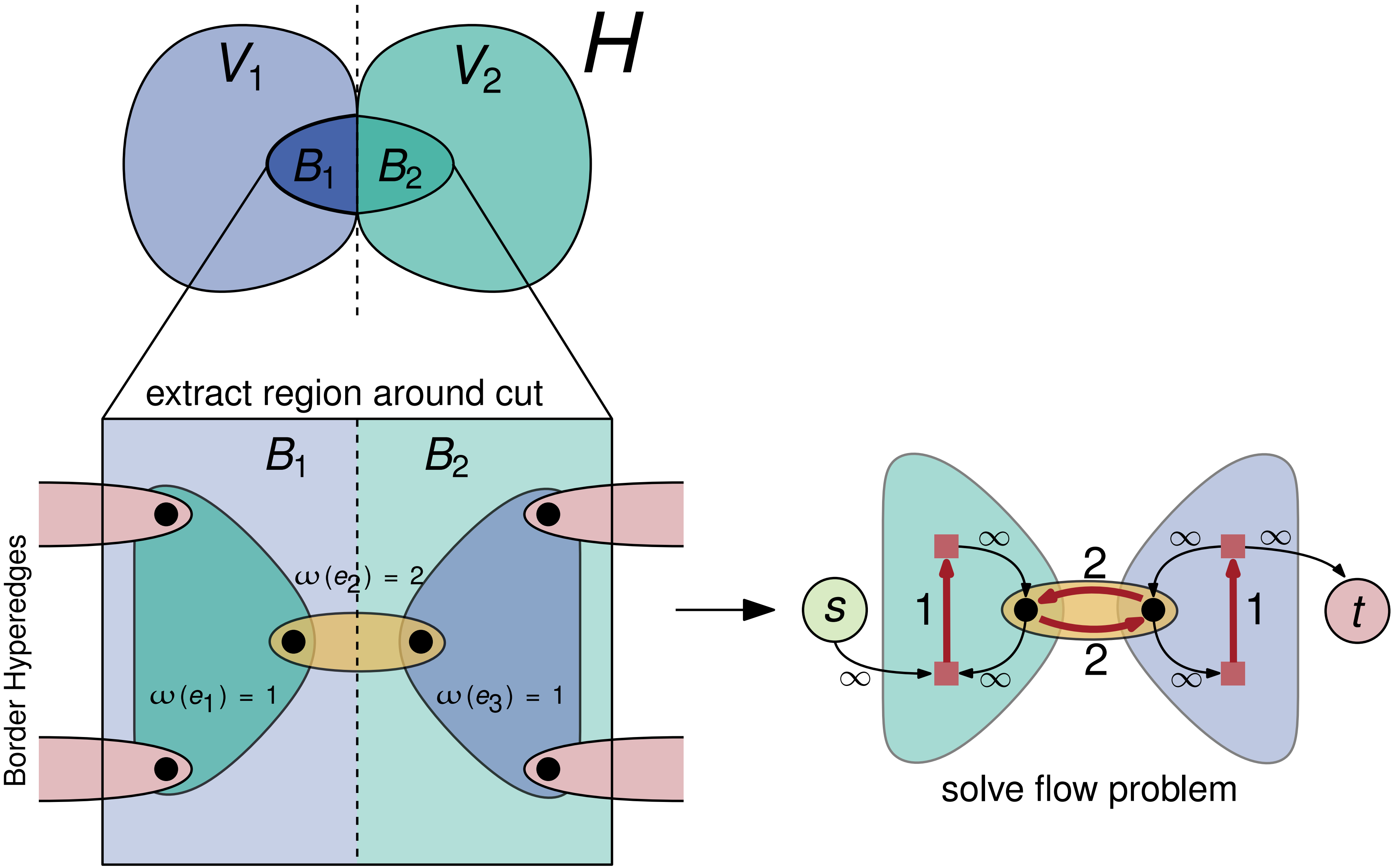
Max-Flow Min-Cut Refinement



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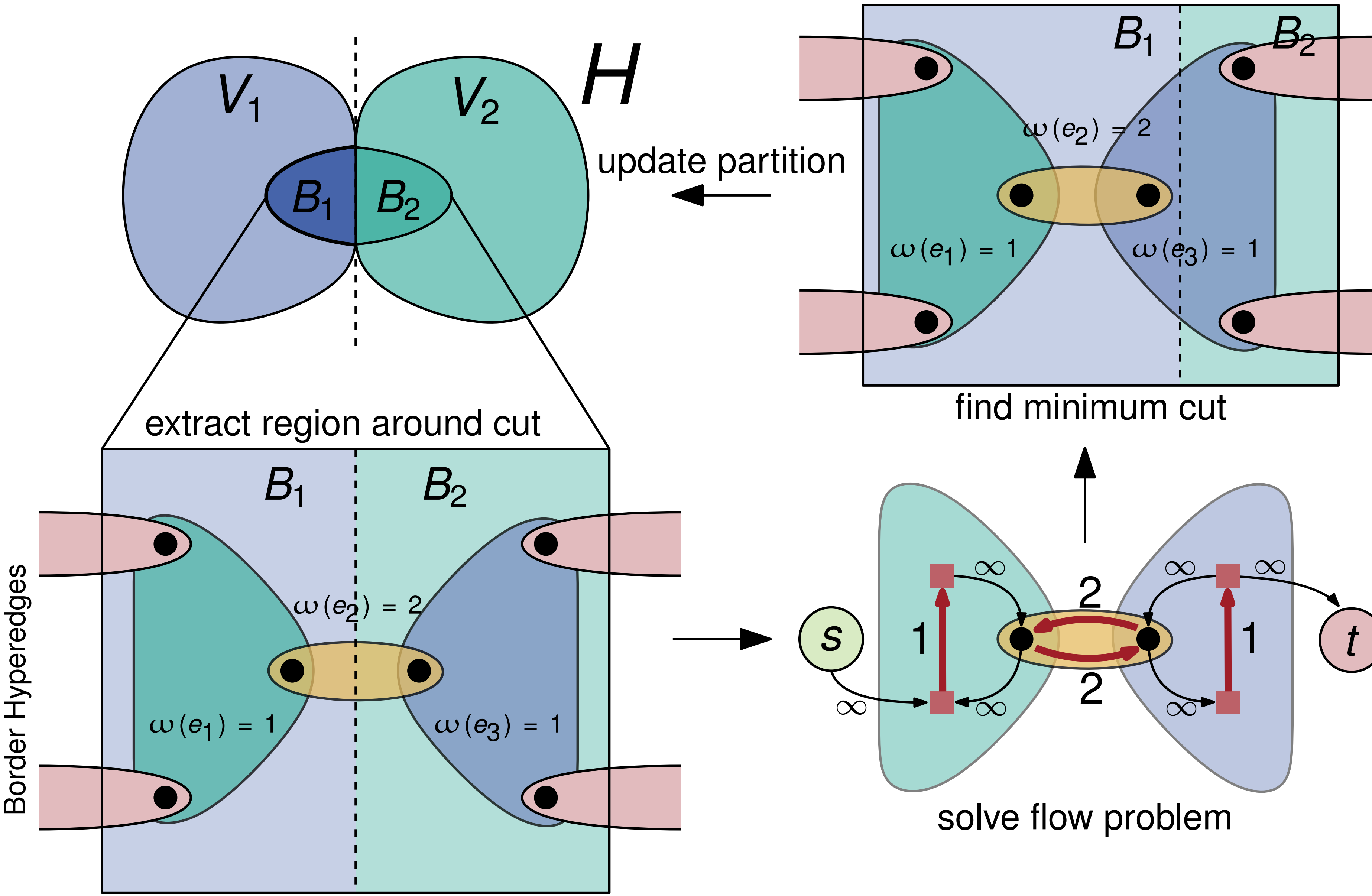
Max-Flow Min-Cut Refinement



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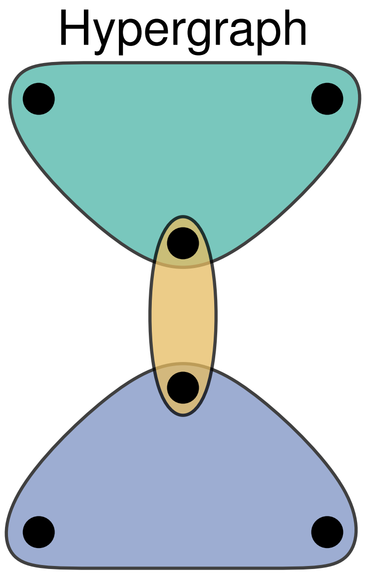
From Hypergraphs to Flow-Networks



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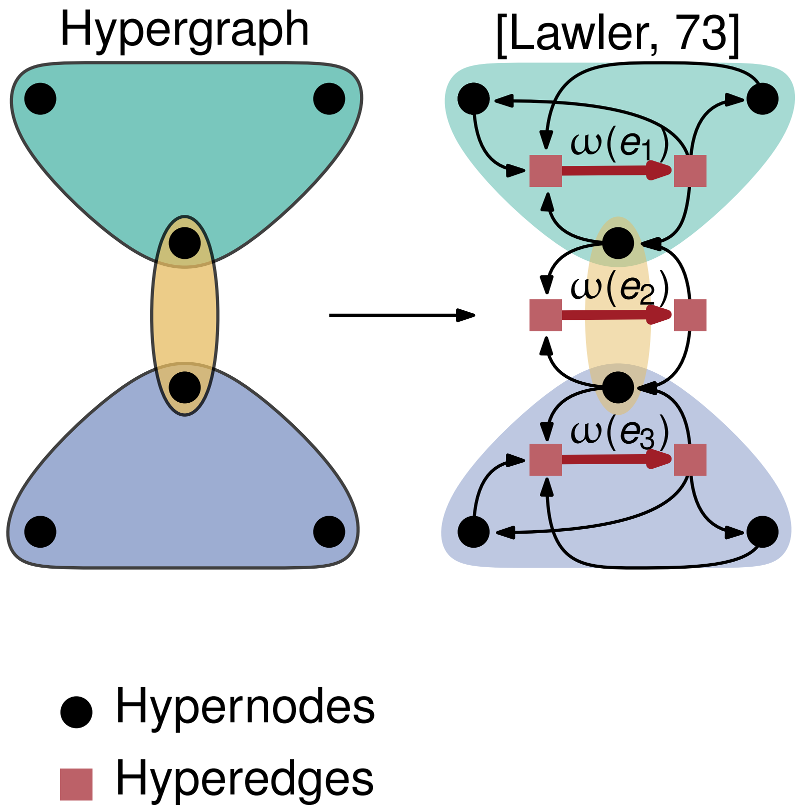
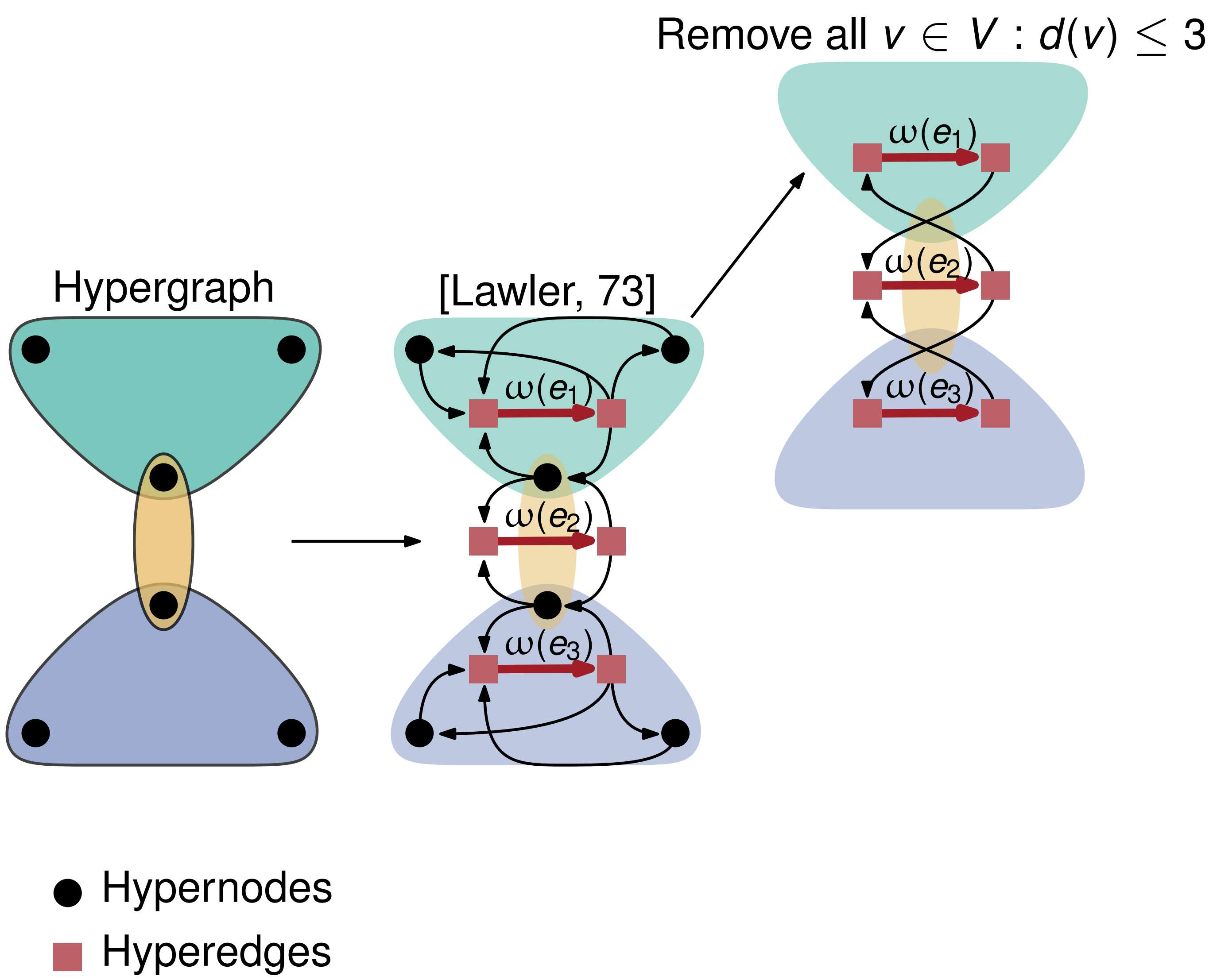
From Hypergraphs to Flow-Networks



From Hypergraphs to Flow-Networks



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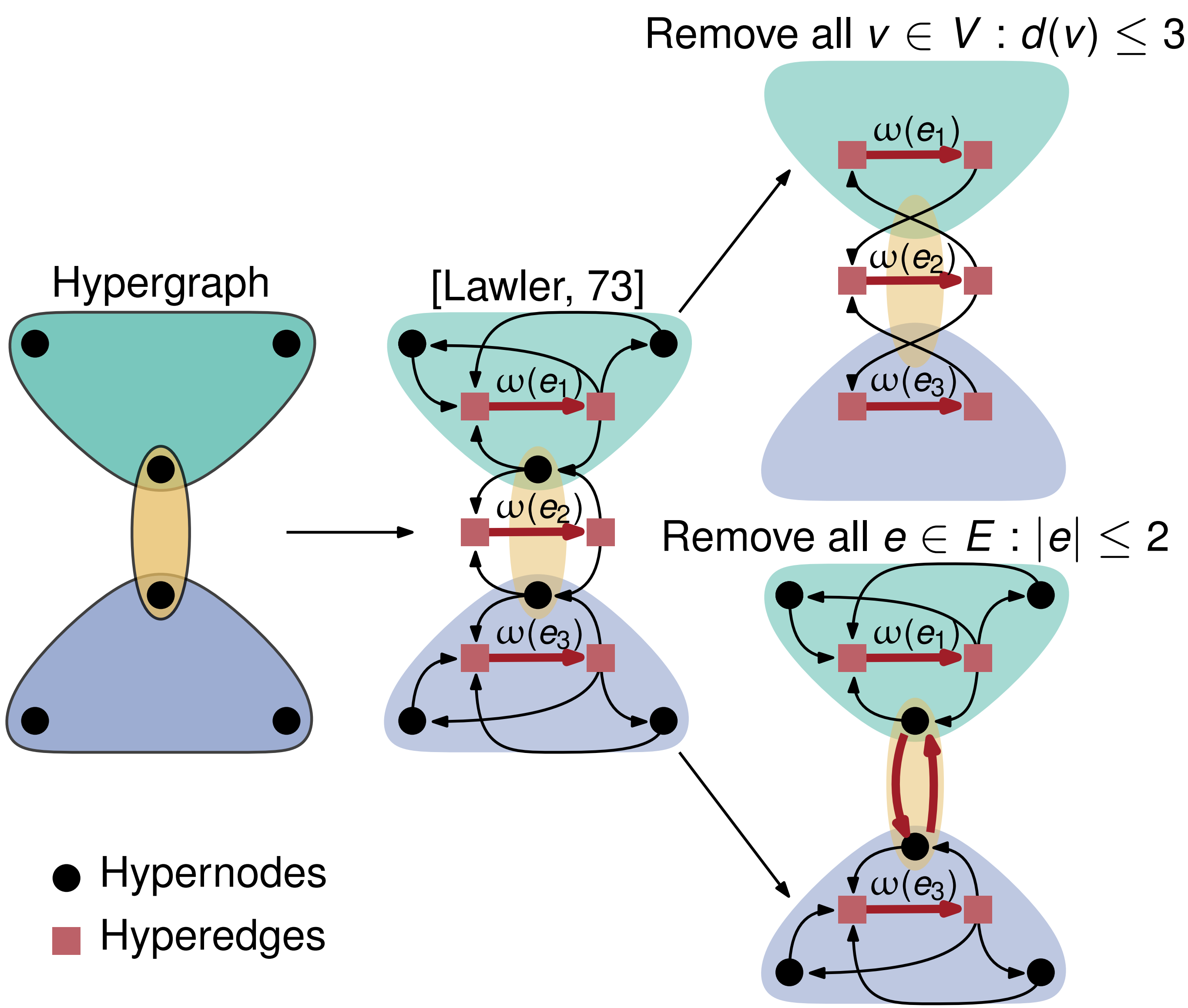

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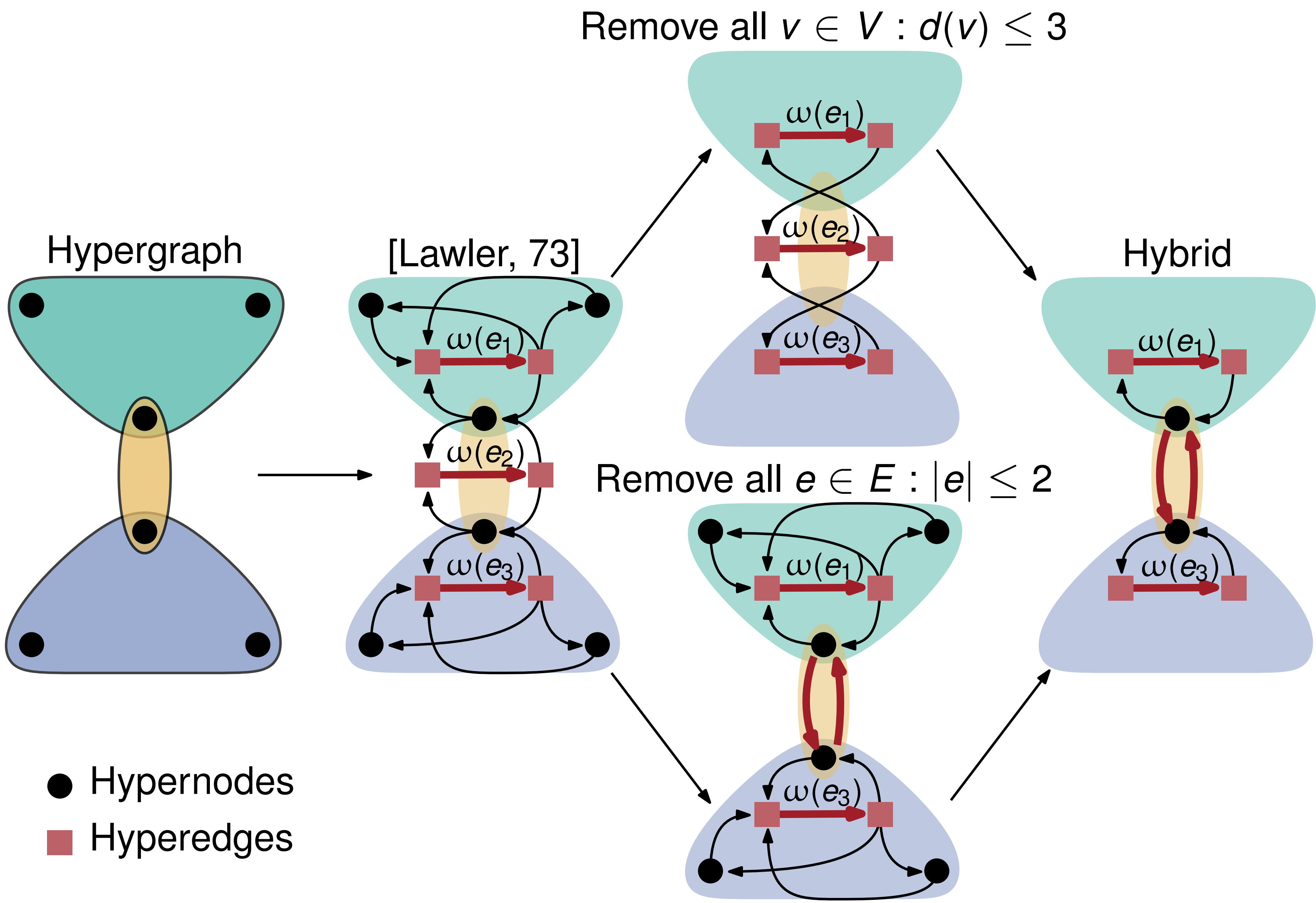

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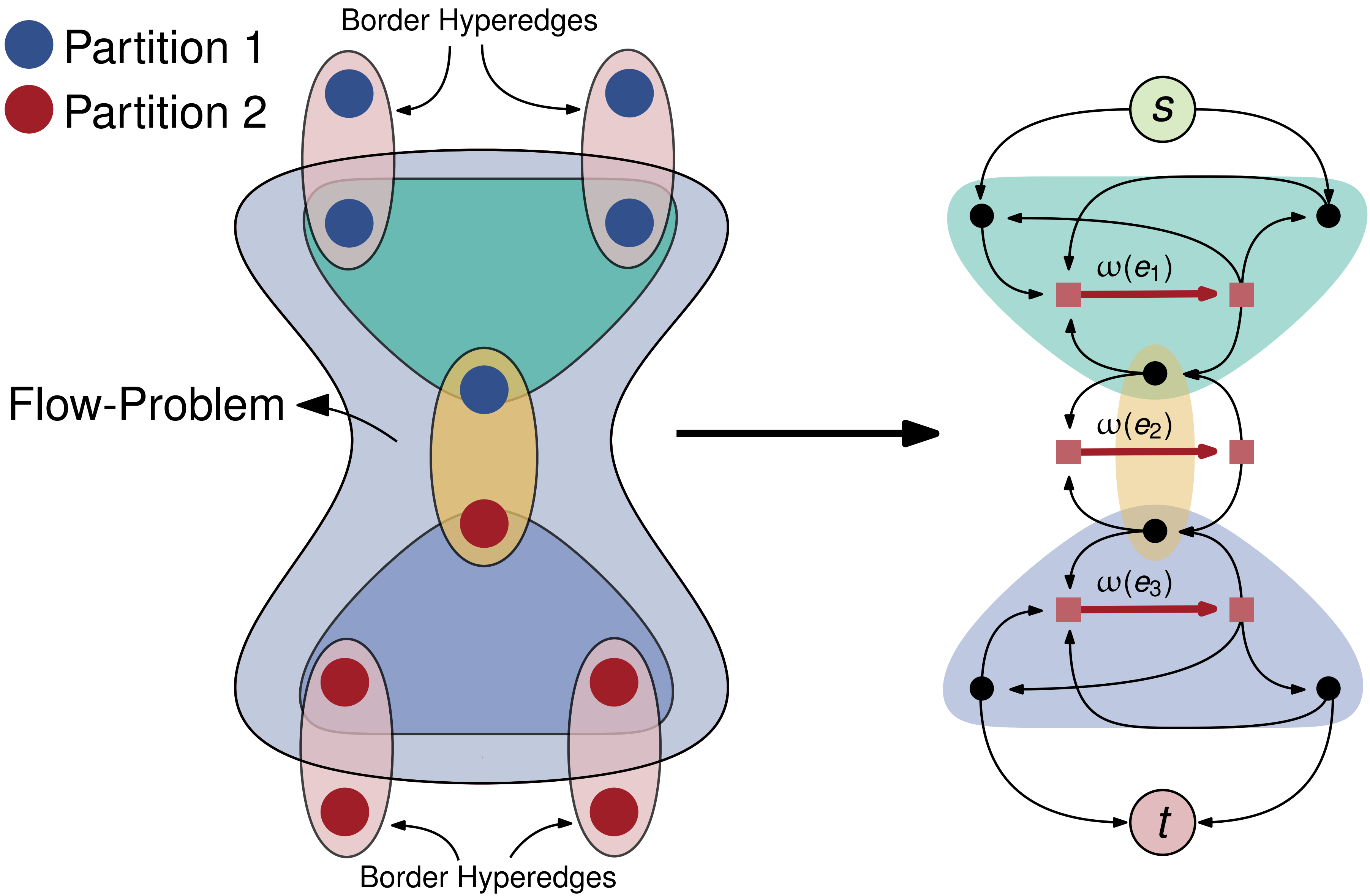
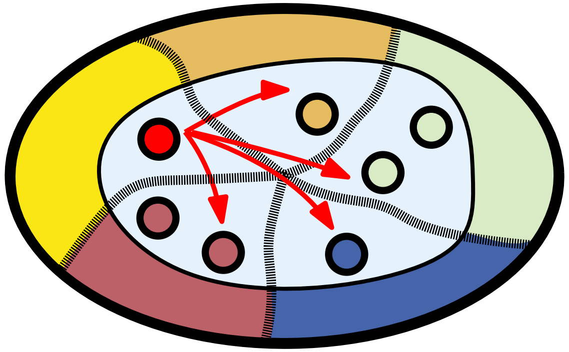
Flow Refinement - Modeling Details



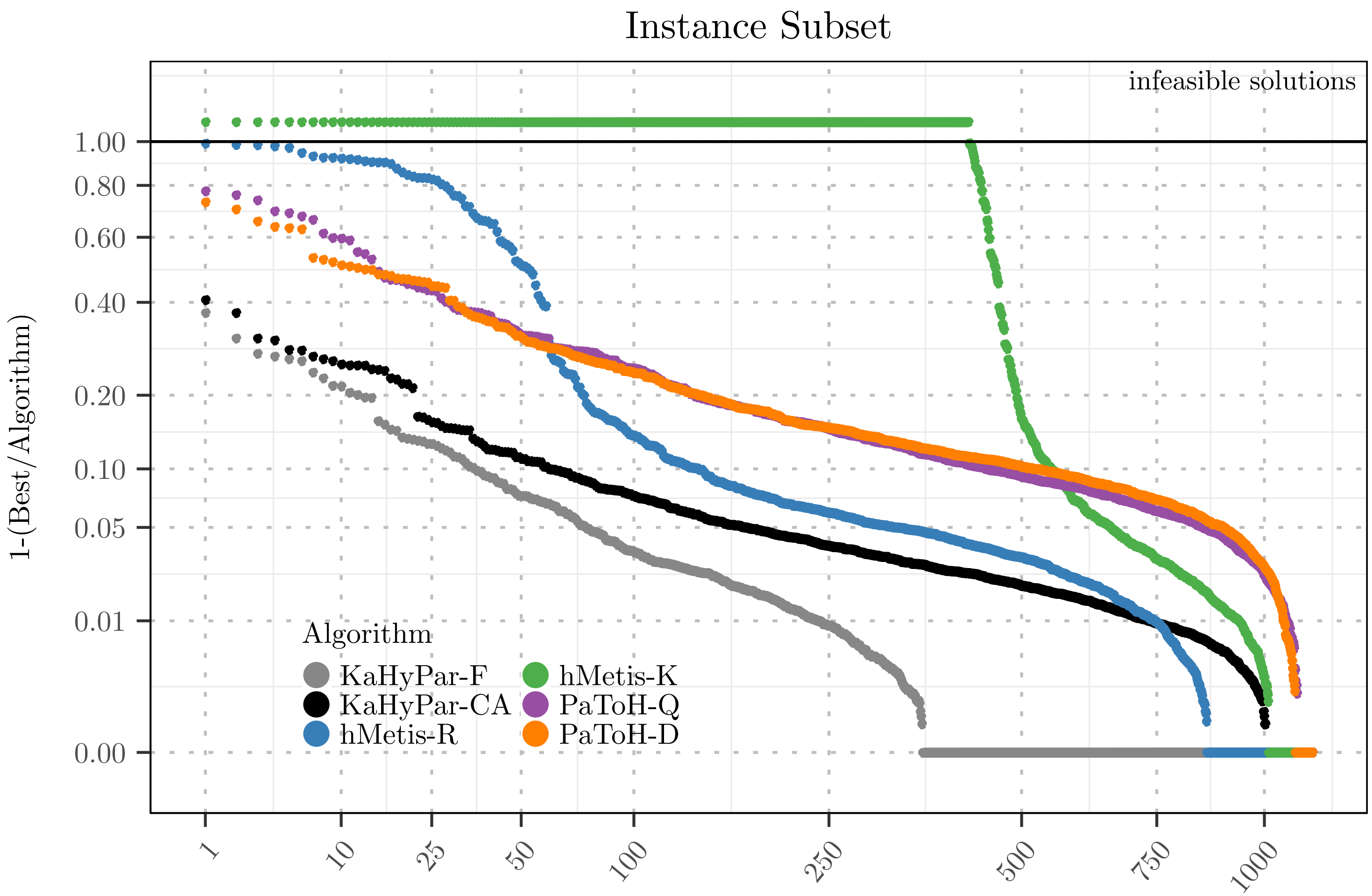
Implementation Details

- *n*-level flow refinement **expensive** $\rightsquigarrow$ emulate multilevel-approach
  $\Rightarrow$ employ every $\frac{n}{\log n}$ uncontractions
- direct *k*-way optimization via **pairwise** flow refinement
- active block scheduling
- most-balanced minimum cuts
- combined with direct *k*-way localized FM local search (every level)



direct *k*-way FM

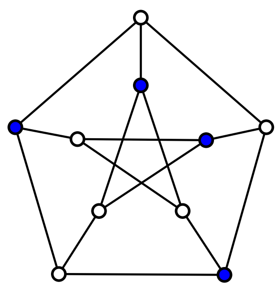
Flow Refinement - Preliminary Results



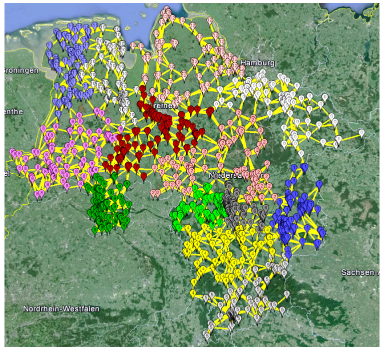
Algorithm	$t[s]$
KaHyPar-CA	21.0
KaHyPar-F	46.8
hMetis-R	63.1

Recent Advances in Graph Partitioning

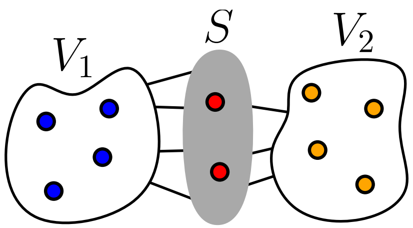
Applications:



Indepentent Sets
Lamm, Sanders, Schulz (SEA'15)
Lamm et. al (ALENEX'16)



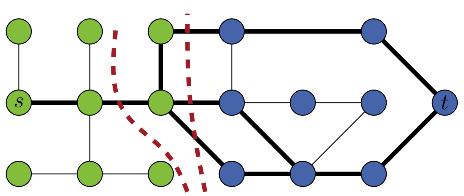
Territory Design
Ahuja et. al (GIS'15)



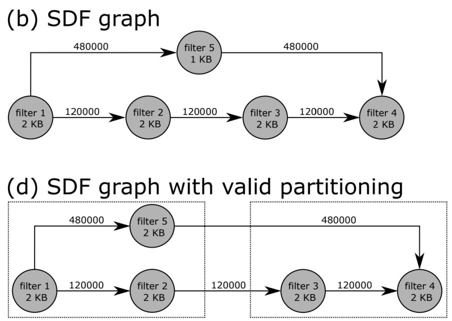
Node Separators
Sanders, Schulz (SEA'16)
Schulz et. al (GECCO'17)

Process Mapping
Schulz, Träff (SEA'17)

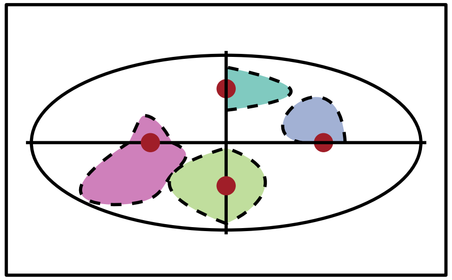
Algorithms:



Flow-Cutter
Hamann, Strasser (ALENEX'16)
Strasser (PACE'16, PACE'17)



DAG Partitioning
Moreia, Popp, Schulz (SEA'17)

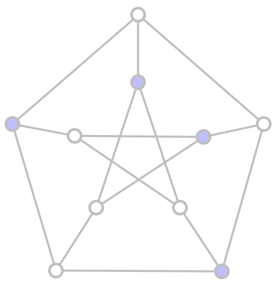


Shared Memory Parallel
Akhremtsev, Sanders, Schulz

Recent Advances in Graph Partitioning

Karlsruhe Institute of Technology

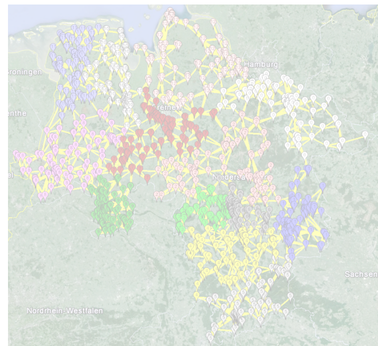
Applications:



Indepentent Sets

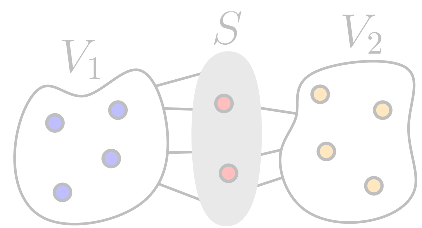
Lamm, Sanders, Schulz (SEA'15)

Lamm et. al (ALENEX'16)



Territory Design

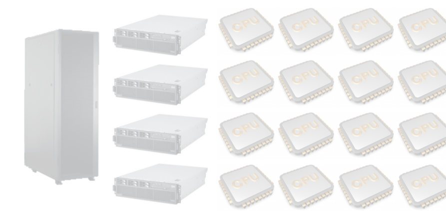
Ahuja et. al (GIS'15)



Node Separators

Sanders, Schulz (SEA'16)

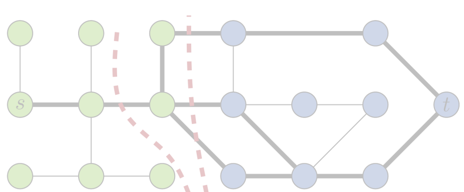
Schulz et. al (GECCO'17)



Process Mapping

Schulz, Träff (SEA'17)

Algorithms:



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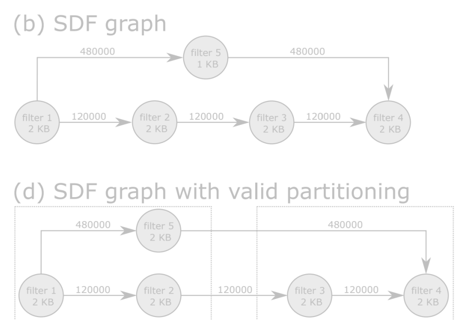

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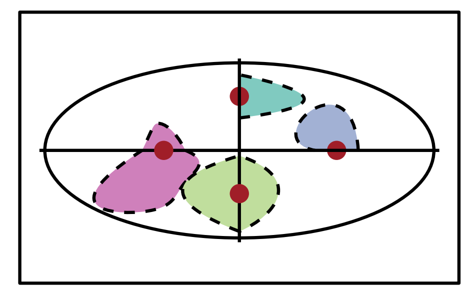


Shared Memory Parallel

Akhremtsev, Sanders, Schulz

Institute of Theoretical Informatics
Algorithmics Group

Parallelizing KaHIP

Goal: High-quality Shared-Memory Graph Partitioner

- **parallel** coarsening, initial partitioning and refinement phases
- constructs balanced solutions of high quality
- good speed-ups, especially for local search algorithms

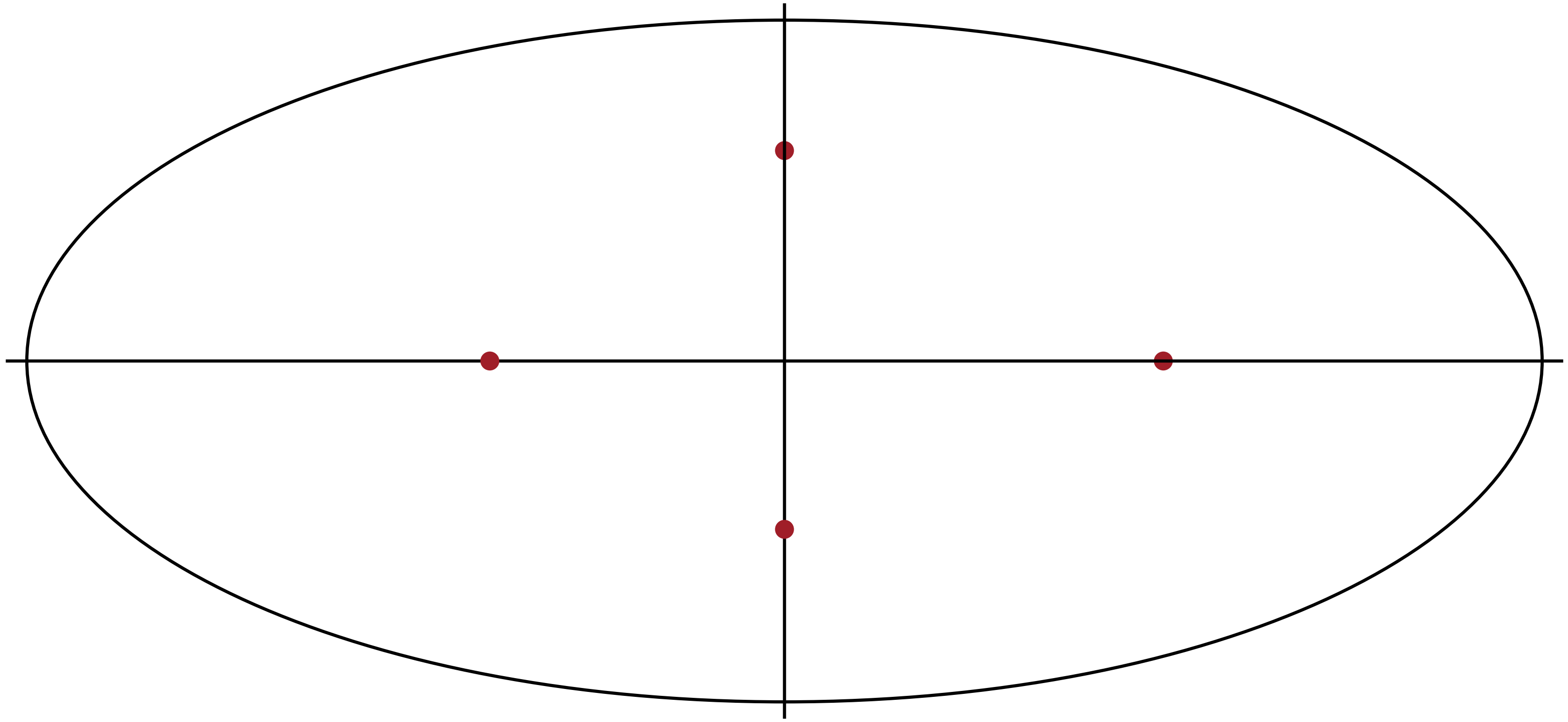
Implementation Details:

- cache-aligned arrays
- TBB scalable allocator
- thread-pinning
- cache-aware hash tables

Slides by Y. Akhremtsev

Parallelizing KaHIP - Parallel Local Search

Each thread chooses a random boundary vertex to start



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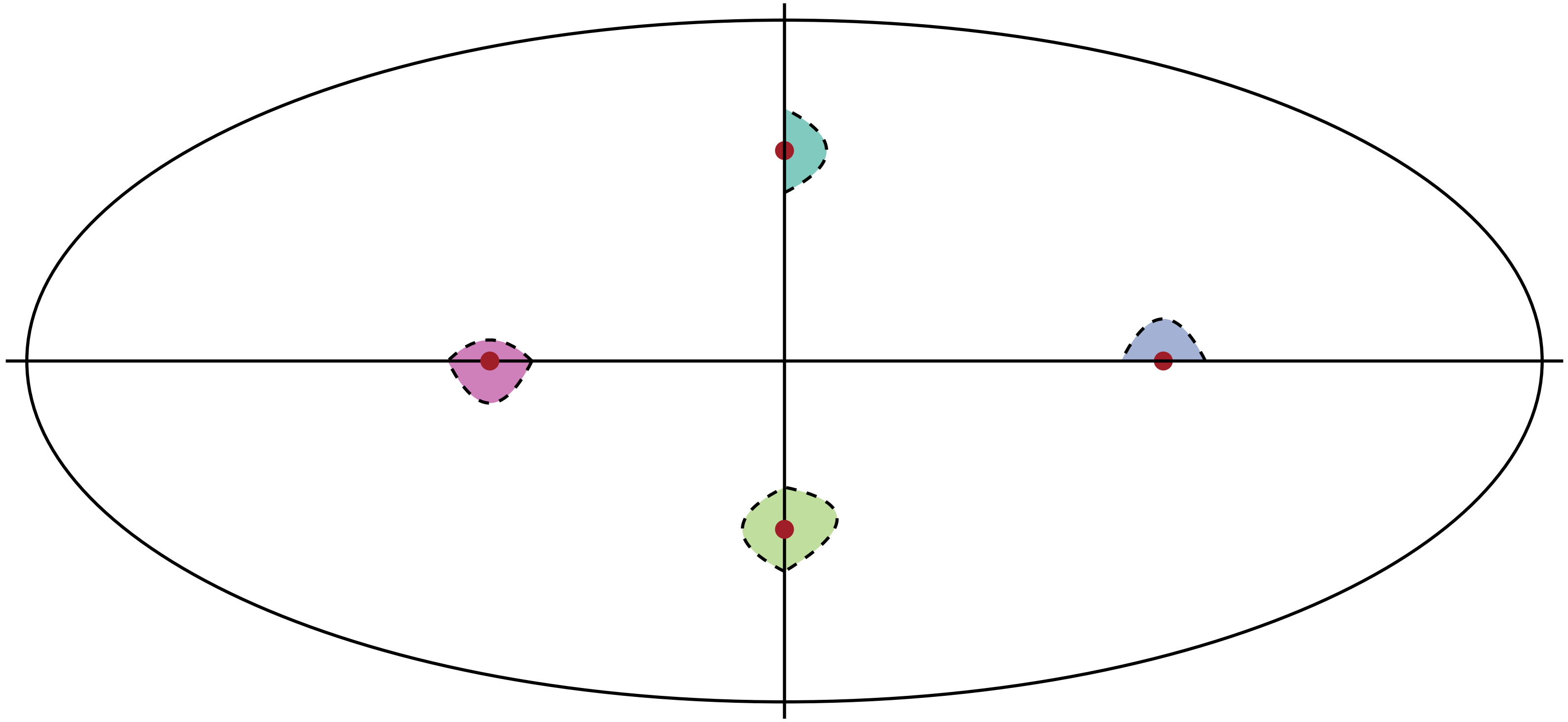
Parallelizing KaHIP - Parallel Local Search

And starts to perform moves around it



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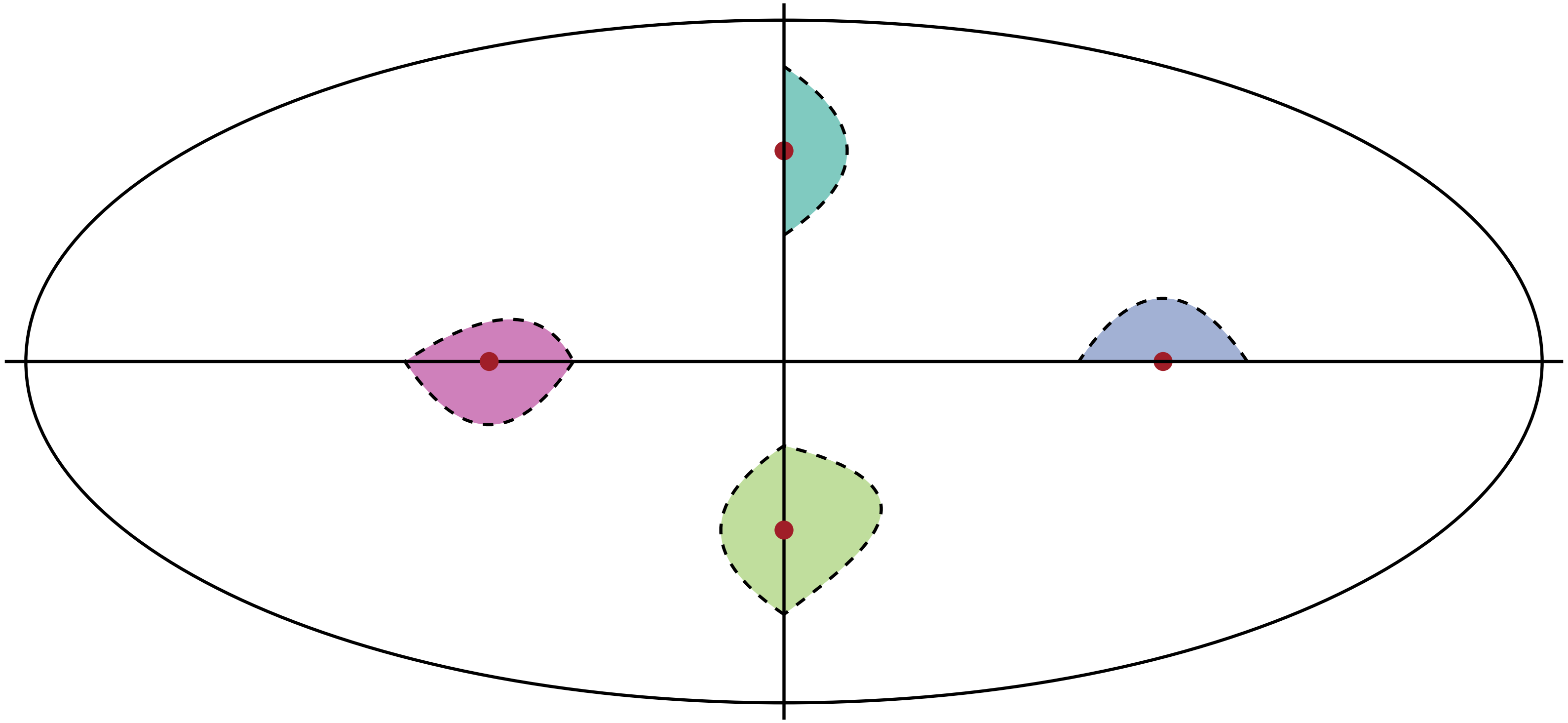
Parallelizing KaHIP - Parallel Local Search

And starts to perform moves around it



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Parallelizing KaHIP - Parallel Local Search

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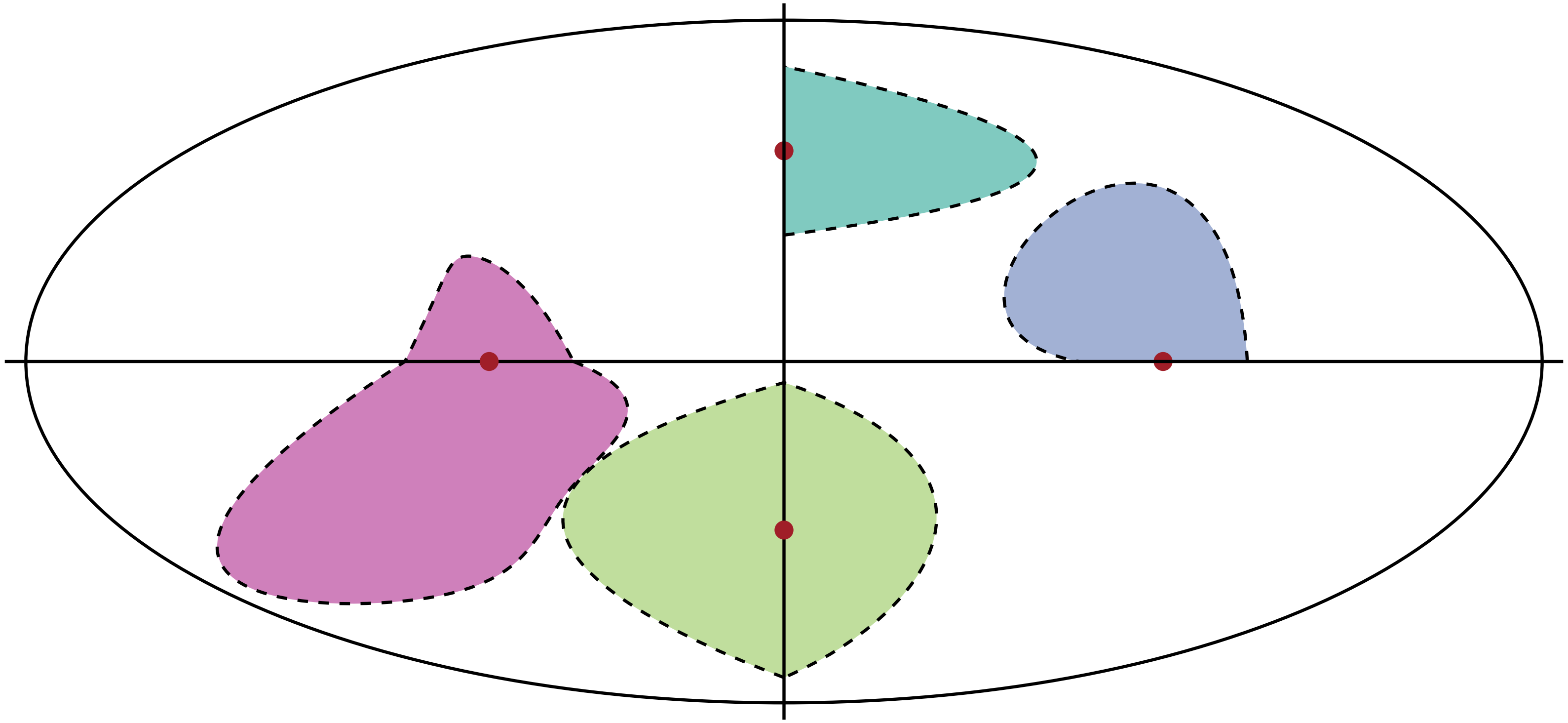
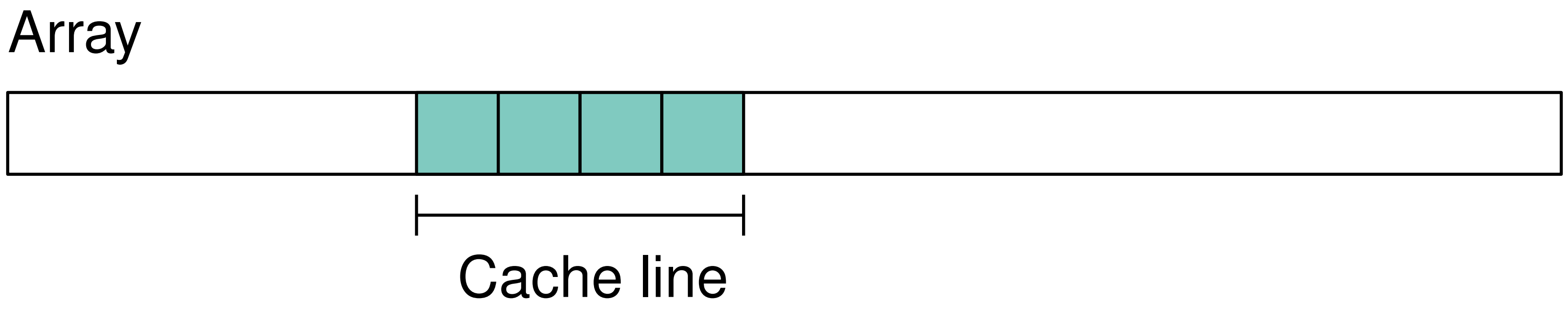


Slides by Y. Akhremtsev

Implementation - Cache-Aware Hash Tables



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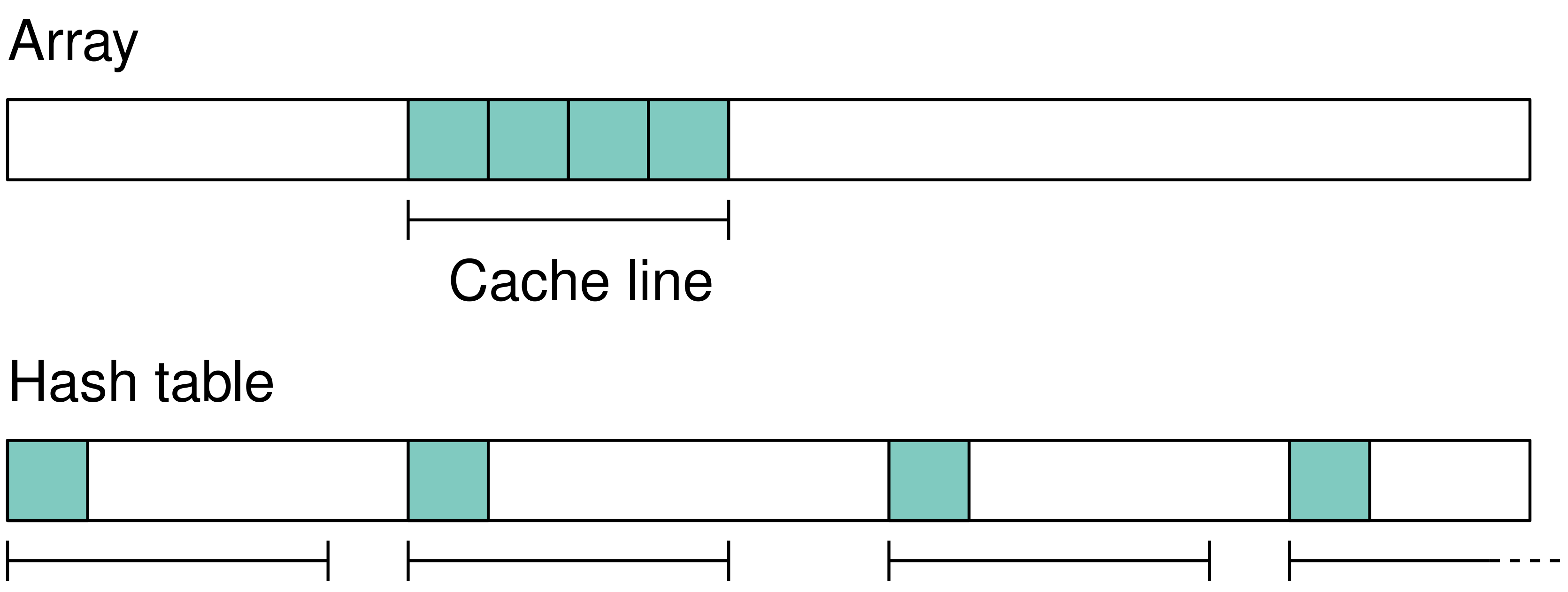

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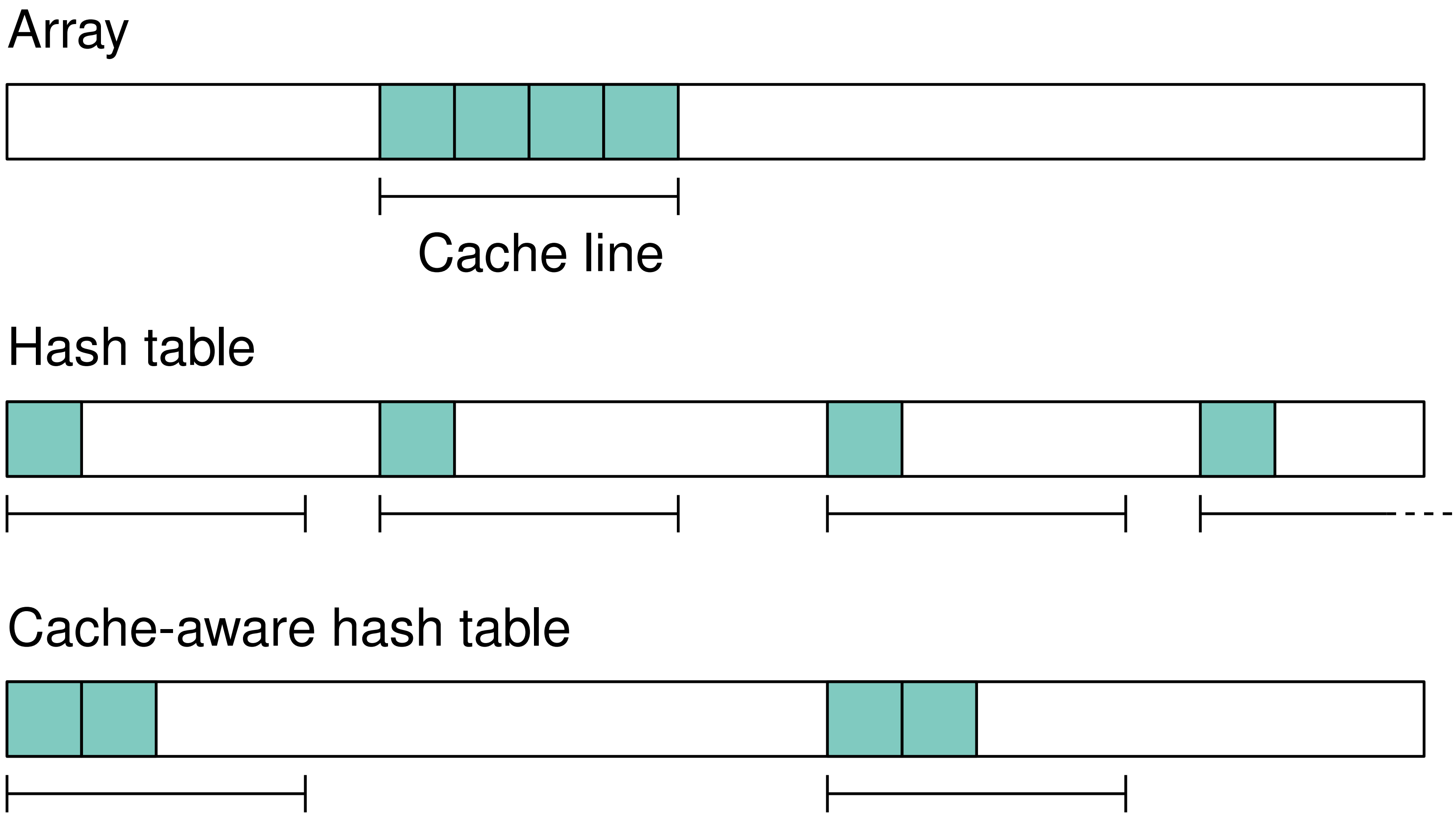

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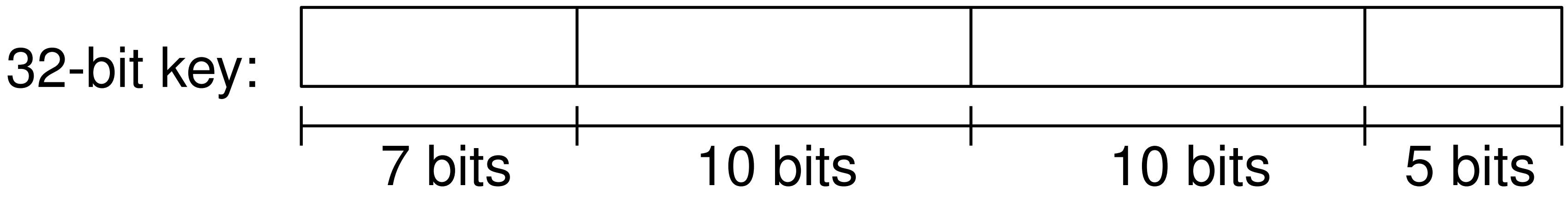
Implementation - Cache-Aware Hash Tables

Tabular Hashing



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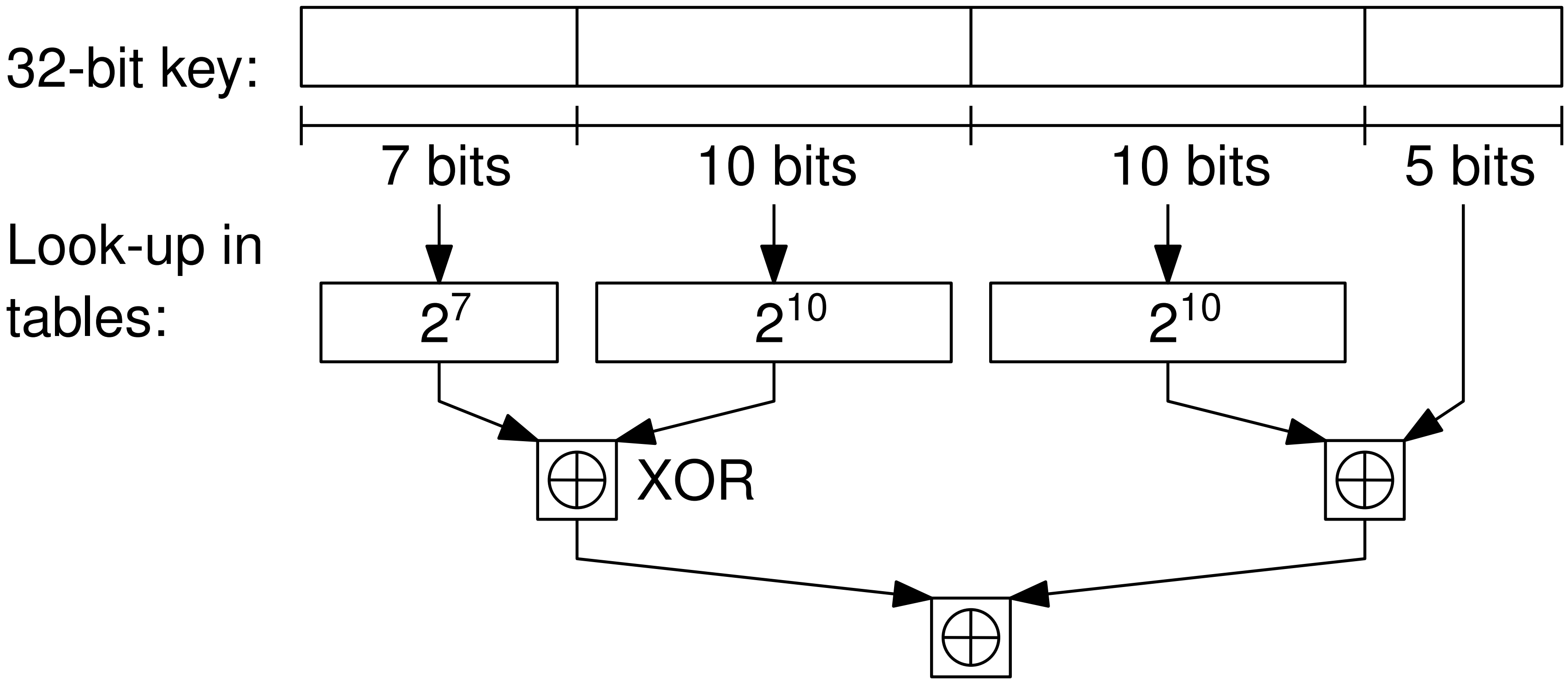
Implementation - Cache-Aware Hash Tables

Tabular Hashing



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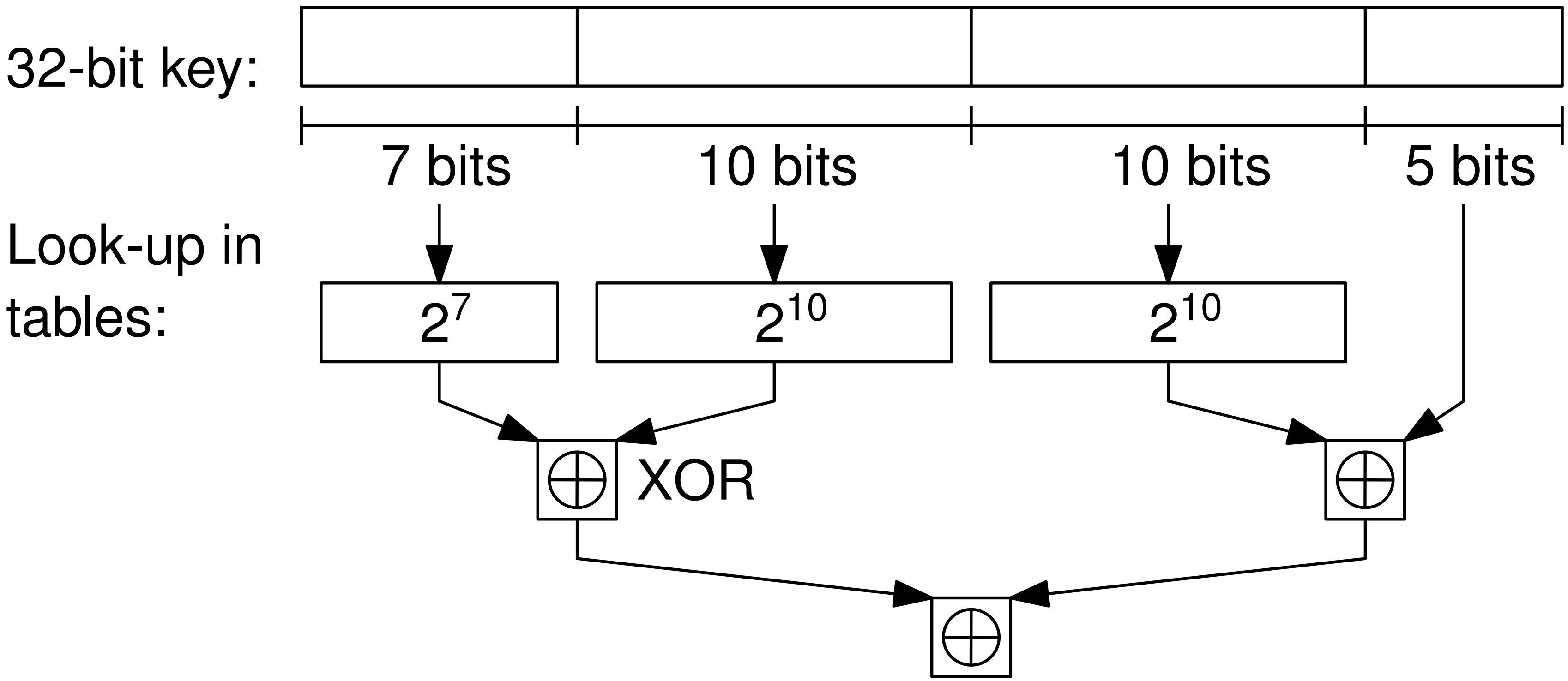
Implementation - Cache-Aware Hash Tables

Tabular Hashing



If $|x - y| \leq 2^5$ then $|h(x) - h(y)| \leq 2^5$

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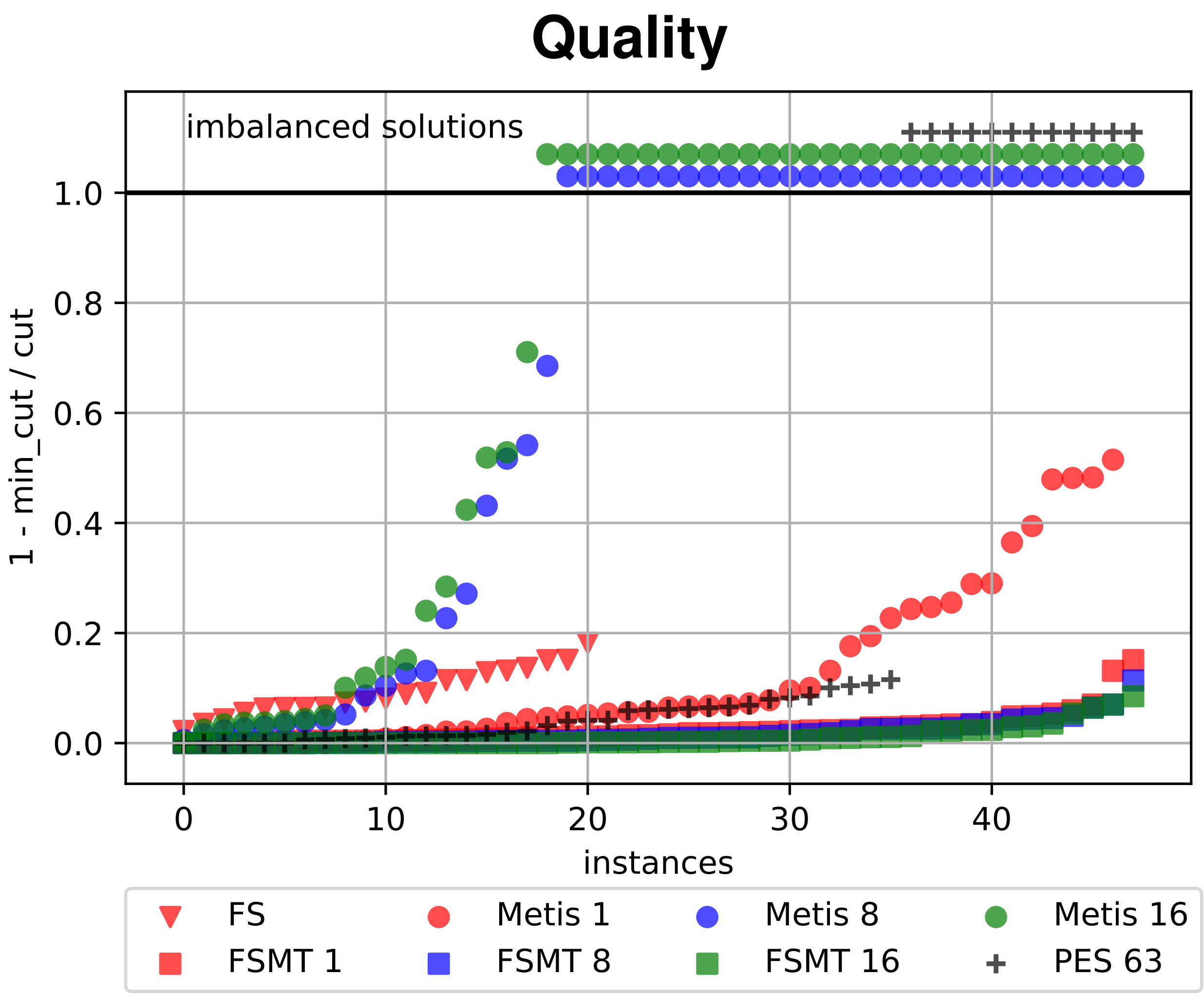
Parallelizing KaHIP - Preliminary Results



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Parallelizing KaHIP - Preliminary Results

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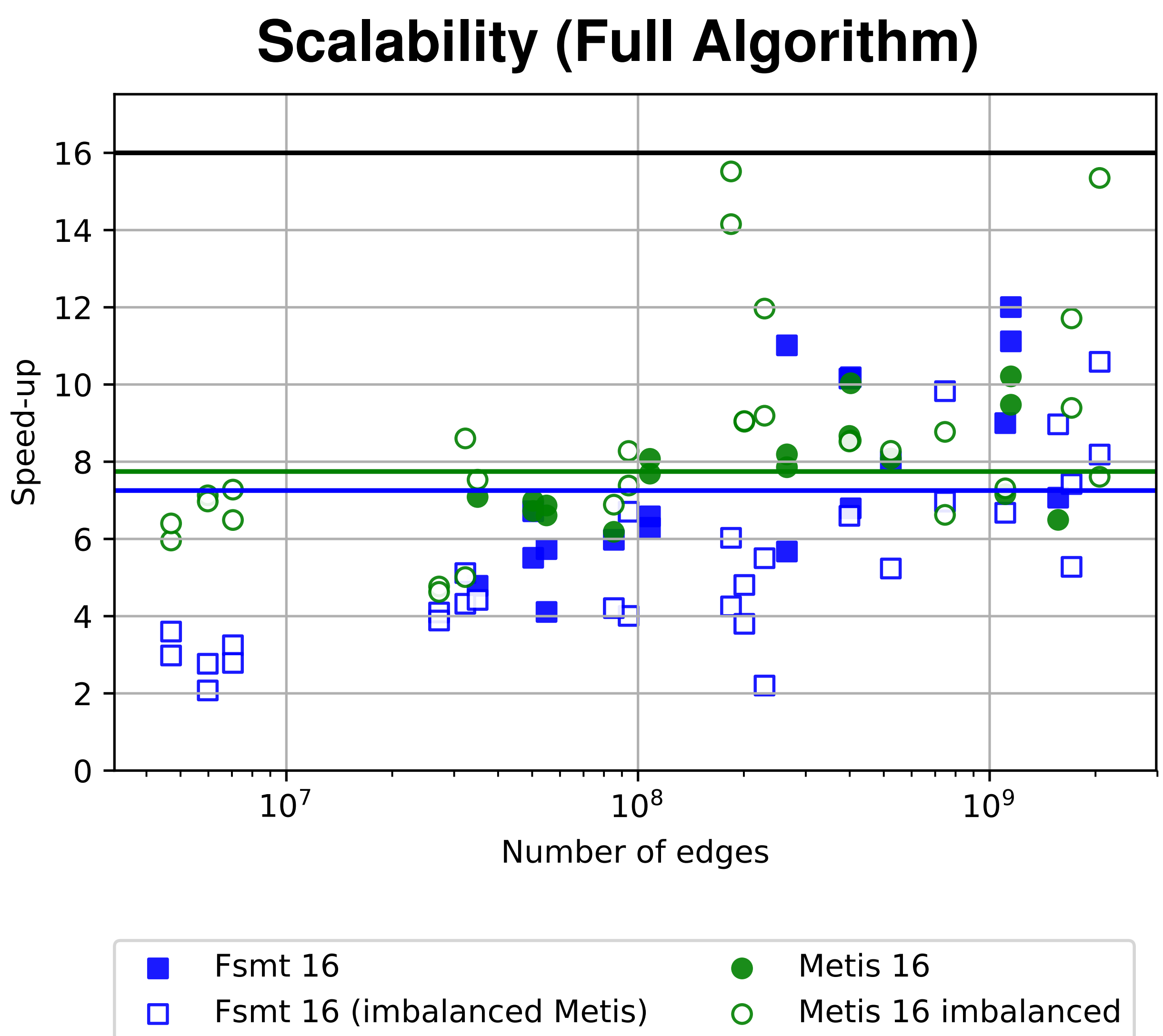
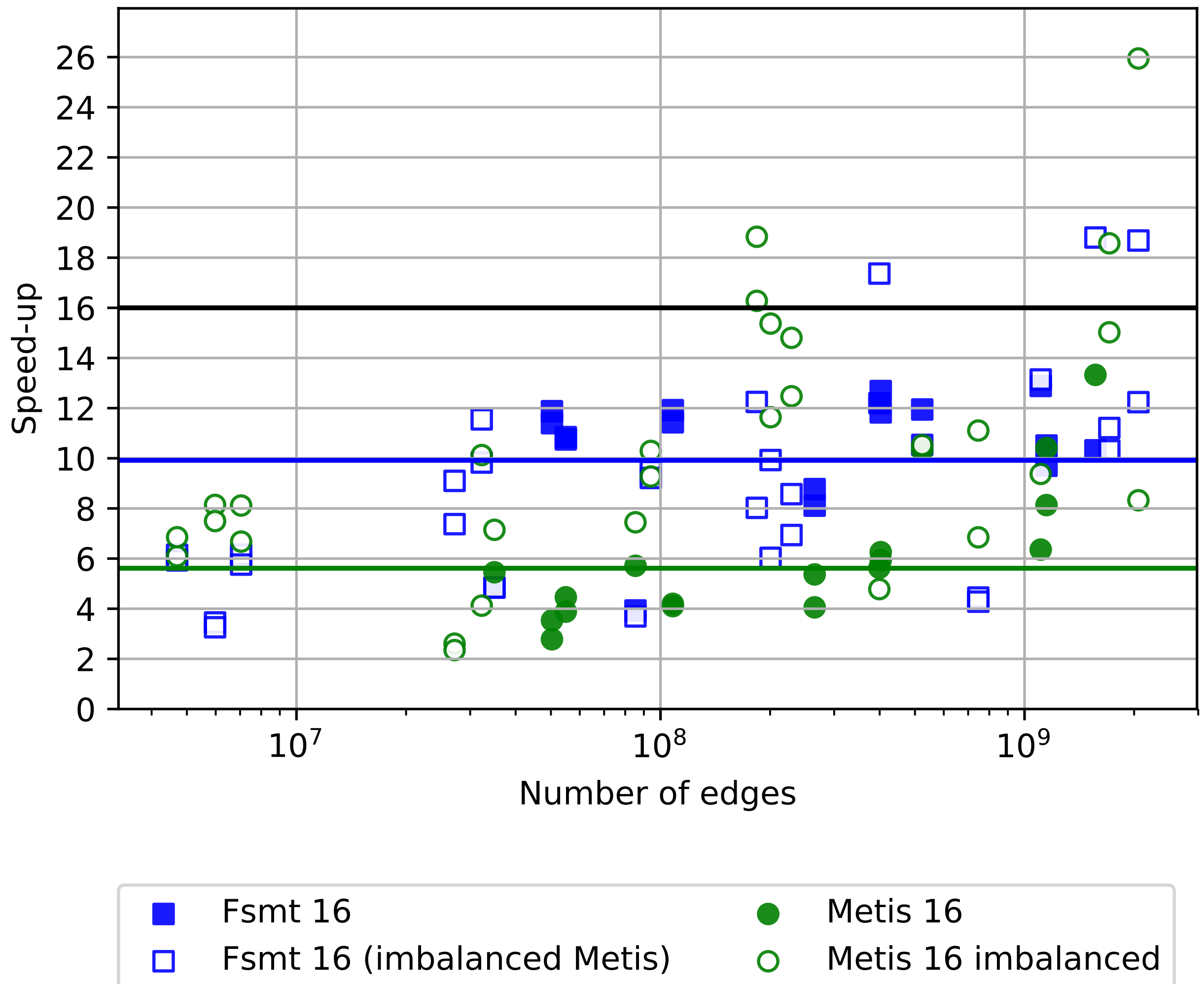


Speed-up for $p = 16$

Slides by Y. Akhremtsev

Parallelizing KaHIP - Preliminary Results

Scalability (Local Search)



Speed-up for p = 16

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Conclusion

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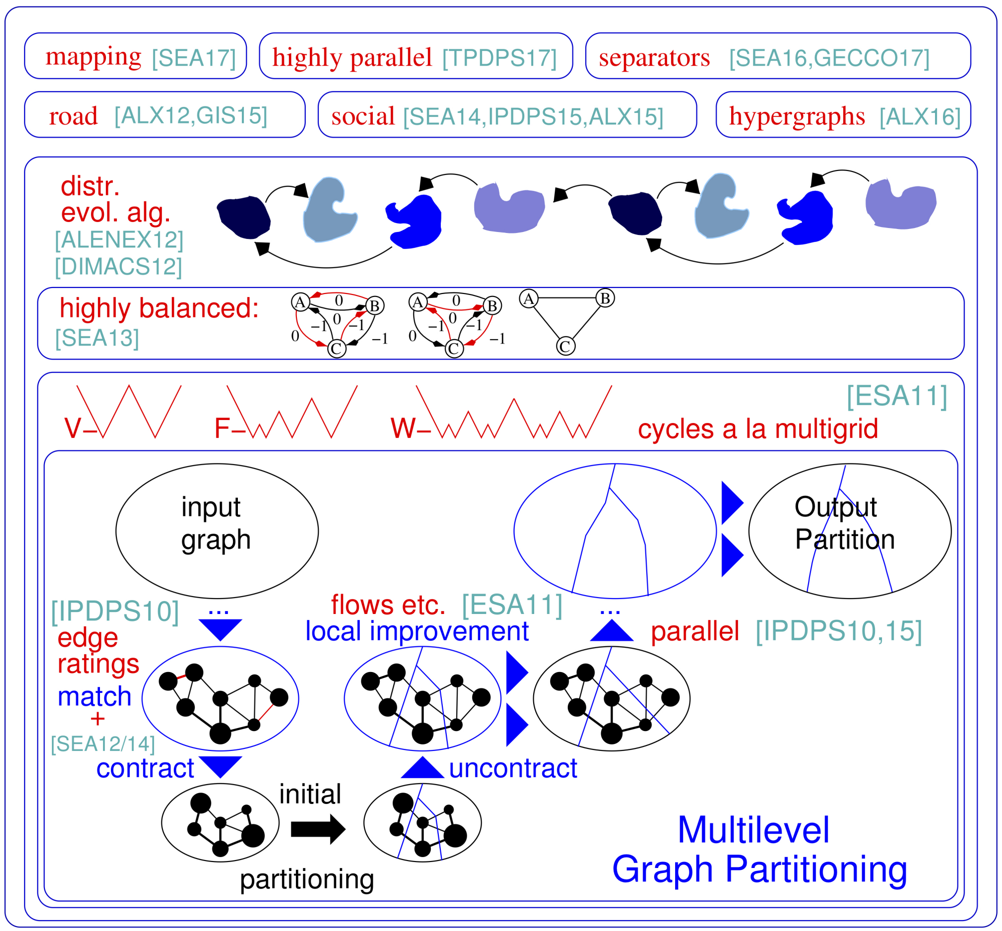
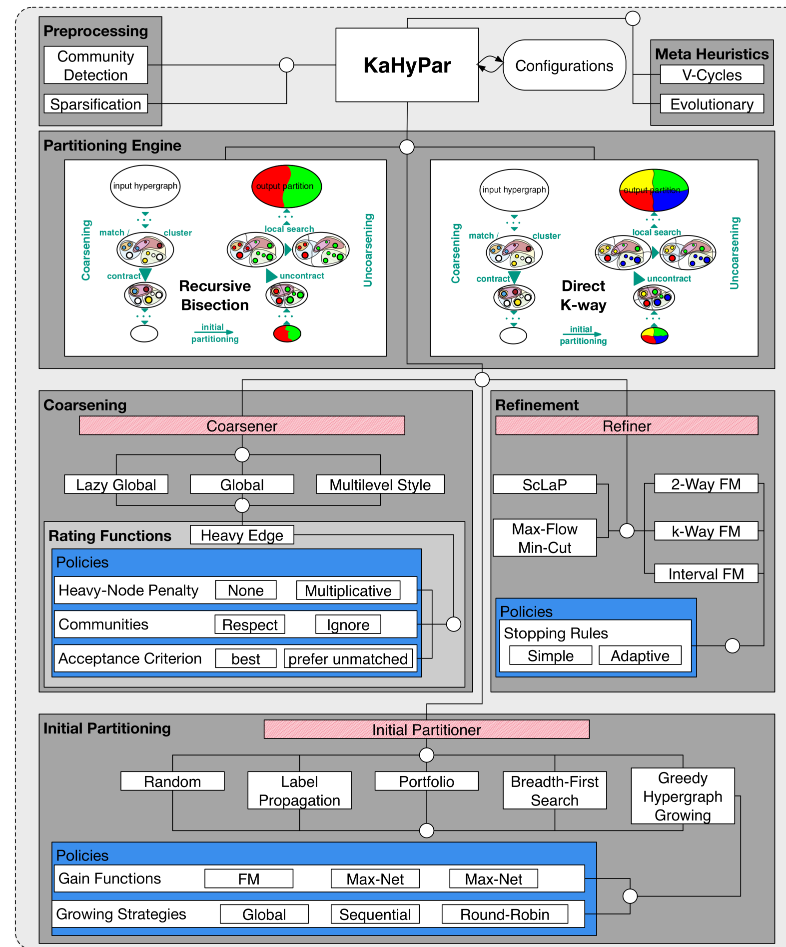
(Hyper)Graph Partitioning Algorithms:

- Graphs: KaHIP – `http://algo2.iti.kit.edu/kahip/`
- Hypergraphs: KaHyPar – `http://www.kahypar.org`





References

Akhremtsev et. al (ALENEX'17): Y. Akhremtsev, T. Heuer, P. Sanders, and S. Schlag. Engineering a direct k-way hypergraph partitioning algorithm. In 19th Workshop on Algorithm Engineering and Experiments, (ALENEX), pages 28–42, 2017.

Heuer, Schlag (SEA'17): T. Heuer and S. Schlag. Improving Coarsening Schemes for Hypergraph Partitioning by Exploiting Community Structure. In 16th International Symposium on Experimental Algorithms, (SEA), page 21:121:19, 2017.

Andre, Schlag, Schulz (arXiv): A., Robin, S. Schlag, and C. Schulz. Memetic Multilevel Hypergraph Partitioning. arXiv preprint arXiv:1710.01968 (2017).

Lamm, Sanders, Schulz (SEA'15): S. Lamm., P. Sanders, C. Schulz. Graph partitioning for independent sets. In 16th International Symposium on Experimental Algorithms, (SEA), page 68-81, 2015.

Lamm et. al (ALENEX'16): Lamm, S., Sanders, P., Schulz, C., Strash, D., Werneck, R. F. (2017). Finding near-optimal independent sets at scale. Journal of Heuristics, 23(4), 207-229.

Ahuja et. al (GIS'15): Ahuja, N., Bender, M., Sanders, P., Schulz, C., Wagner, A. (2015, November). Incorporating road networks into territory design. In Proceedings of the 23rd SIGSPATIAL International Conference on Advances in Geographic Information Systems (p. 4). ACM.

Sanders, Schulz (SEA'16): Sanders, P., Schulz, C. Advanced Multilevel Node Separator Algorithms. In 16th International Symposium on Experimental Algorithms, (SEA), pages 294-309, 2016.

Schulz et. al (GECCO'17): P. Sanders, C. Schulz, D. Strash, R. Williger. 2017. Distributed evolutionary k-way node separators. In Proceedings of the Genetic and Evolutionary Computation Conference (GECCO '17).

Schulz, Träff (SEA'17): C. Schulz and J. Larsson Träff. Better Process Mapping and Sparse Quadratic Assignment. In 16th International Symposium on Experimental Algorithms, (SEA), page 4:1–4:15 2017.

Hamann, Strasser (ALENEX'16): Hamann, M., Strasser, B. (2016). Graph bisection with pareto-optimization. I. In 19th Workshop on Algorithm Engineering and Experiments, (ALENEX), pages 90–102, 2017.

Moreia, Popp, Schulz (SEA'17): O. Moreira, M. Popp, C. Schulz. Graph Partitioning with Acyclicity Constraints. In 16th International Symposium on Experimental Algorithms, (SEA), pages 30:1–30:15, 2017.